On the width and roundness of a set of points in the plane

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Abstract

Let S be a set of points in the plane. The width (resp. roundness) of S is defined as the minimum width of any slab (resp. annulus) that contains all points of S. We give a new characterization of the width of a point set. Also, we give a *rigorous* proof of the fact that either the roundness of S is equal to the width of S, or the center of the minimum-width annulus is a vertex of the closest-point Voronoi diagram of S, the furthest-point Voronoi diagram of S, or an intersection point of these two diagrams. This proof corrects the characterization of roundness used extensively in the literature.

1 Introduction

The problem of approximating point sets by simple geometric figures has received great attention. As an example, assume we have a large number of mass-manufactured circular profiles. In order to test the quality of such a profile, we take sample points from its surface. The profile is acceptable if the smallest annulus that contains all these sample points has a width that is less than some tolerance factor. (The American National Standards Institute recommends this measure to be used for testing circular profiles, see Foster [5, pages 40–42] and Le and Lee [7].)

In this paper, we consider two such approximation problems. Before we can formulate them, we need some definitions. A *slab* is defined as the closed region lying between any two parallel lines in the plane, and an *annulus* as the closed region lying between any two concentric circles of finite radius in the plane. The *width* of a slab (resp. annulus) is defined as the distance between its bounding lines (resp. the difference of the radii of its bounding circles).

Problem 1 *Given a set S of points in the plane, compute its* width, *which is defined as the minimum width of any slab that contains all points of S.*

A slab whose width is equal to the width of the point set and that contains all points of S will be referred to as an *optimal slab.*

Problem 2 *Given a set S of points in the plane, compute its* roundness, *which is defined as the minimum width of any annulus that contains all points of S.*

An annulus whose width is equal to the roundness of the point set and that contains all points of S will be referred to as an *optimal annulus.*

Problem 1 was considered by Houle and Toussaint [6]. They showed that there is an optimal slab that is bounded by an antipodal pair consisting of a vertex and an edge of the convex hull of S. Using this characterization, they derived an $O(n \log n)$ time algorithm for computing the width of a set of n points.

In this paper, we give a new characterization of the width of a planar point set. This leads to an $O(n \log n)$ time algorithm for computing the width, which is similar to that of Houle and Toussaint.

Problem 2 has received considerable attention recently, see [1, 2, 3, 4, 7, 8]. For $x \in \mathbb{R}^2$, let $N(x)$ (resp. $F(x)$) denote a nearest (resp. furthest) neighbor of x in S. Then the minimum-width annulus centered at x that contains all points of S has width $d(x, F(x)) - d(x, N(x))$, where $d(\cdot,\cdot)$ is the Euclidean distance function. Hence, the problem is to minimize the function $d(x, F(x)) - d(x, N(x))$ over all points x in the plane.

Let $NVD(S)$ (resp. $FVD(S)$) denote the closest-point (resp. furthest-point) Voronoi diagram of S. Let G be the subdivision obtained by superimposing these two diagrams. Hence, the vertex set of G is the union of the vertices of $NVD(S)$, the vertices of $FVD(S)$, and the intersections of edges of these two diagrams. In

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[1, 2, 3, 4], it is claimed that there is an optimal annulus having its center at a vertex of G.

In this paper, we give the correct form of this claim and give a *rigorous* proof.

Efficient algorithms for computing the roundness have been given in [1, 2, 3, 4]; the best known running time is $O(n^{3/2+\epsilon})$, given in [2], where $\epsilon > 0$ is an arbitrarily small constant. We note that our new characterization of the optimal annulus does not affect these bounds.

The main difficulty in the proof is that the minimum of the function $d(x, F(x)) - d(x, N(x))$ may not exist at all and instead we need to characterize its *infimum*. For example, consider a set of points on the Y-axis. For any point x in the plane, the value of $d(x, F(x)) - d(x, N(x))$ is positive. If we let x go to infinity on the positive X-axis, then $d(x, F(x)) - d(x, N(x))$ converges to zero. The reader may argue that this only happens if all points are on a line. This is, however, not the case. In Section 3.5, we give an example of a set of points, not all on a line, such that *any* annulus containing all points has width strictly larger than one, whereas the points are contained in a slab of width one. As it turns out, this fact is one of the reasons why the proof of our main result is non-trivial. It follows that we must reformulate Problem 2 as follows.

Problem 3 *Given a set S of points in the plane, compute its roundness $rd(S)$, defined as*

$$rd(S) := \inf\{d(x, F(x)) - d(x, N(x)) : x \in \mathbb{R}^2\}.$$

1.1 Summary of results

Let S be a set of points in the plane. Let G be the subdivision obtained by superimposing the closest-point Voronoi diagram $NVD(S)$ and the furthest-point Voronoi diagram $FVD(S)$. The vertex set of G is the union of the vertex set of $NVD(S)$, the vertex set of $FVD(S)$, and the set of intersection points of non-overlapping edge pairs (e, e'), where $e \in NVD(S)$ and $e' \in FVD(S)$. We assume that edges are closed.

If e is an unbounded edge of $NVD(S)$, $FVD(S)$, or G, then $w(e)$ will denote the limit of $d(x, F(x)) - d(x, N(x))$, if x goes to infinity on e.

Theorem 1

1. *The width of S is equal to the minimum value of $w(e)$ over all unbounded edges e of $NVD(S)$.*
2. *The width of S is equal to the minimum value of $w(e)$ over all unbounded edges e of $FVD(S)$.*

Theorem 2 *Let $width(S)$ denote the width of S.*

1. *$rd(S) = \min(width(S), \min\{d(v, F(v)) - d(v, N(v)) : v$ is a vertex of $G\})$.*
2. *Suppose there is an optimal slab such that one of its bounding lines contains exactly one point of S. Then $rd(S) < width(S)$.*

2 Characterizing the width of a planar point set

Let S be a set of points in the plane, and let e be a bounded or unbounded edge of $NVD(S)$. Then, each point x in the interior of e has exactly two nearest neighbors. In fact, these neighbors are the same for each such point x. We call each of these neighbors a *nearest neighbor of e*. The *furthest neighbors* of an edge of $FVD(S)$ are defined in a similar way.

Consider an edge e of the subdivision G. If e is (part of) both an $NVD(S)$-edge and an $FVD(S)$-edge, then this edge has two nearest neighbors and two furthest neighbors. If e is (part of) an $NVD(S)$-edge but not (part of) an $FVD(S)$-edge, then this edge has two nearest neighbors and, since the interior of e lies completely inside a face of $FVD(S)$, e has exactly one furthest neighbor. Similarly, if e is (part of) an $FVD(S)$-edge but not (part of) an $NVD(S)$-edge, then e has two furthest neighbors and one nearest neighbor.

Lemma 1 *Let e be an unbounded edge of G. Let p (resp. q) be a nearest (resp. furthest) neighbor of e. Let δ be the distance between the orthogonal projections of p and q onto e. Then, for $v \in e$, the function $d(v, F(v)) - d(v, N(v))$ converges monotonically to δ if v moves along e to infinity. That is, using the notation of Section 1.1, we have $w(e) = \delta$.*

Proof: We assume w.l.o.g. that e is contained in the X-axis, and that it is unbounded to the right. Let p (resp. q) have coordinates (p_1, p_2) (resp. (q_1, q_2)). Note that $p_1 \geq q_1$, because otherwise, there is a point on e far to the right that is closer to q than to p. Consider the function $f(x) := \sqrt{(x-q_1)^2 + q_2^2} - \sqrt{(x-p_1)^2 + p_2^2}$, for real numbers x such that $(x, 0)$ is a point of e. Hence, $f(x)$ is the distance between $(x, 0)$ and its furthest neighbor minus the distance between $(x, 0)$ and its nearest neighbor.

We analyze the behavior of f for large x. Using the asymptotic[1] expansion $\sqrt{1+h} = 1 + h/2 + O(h^2)$, $(h \to 0)$, we get $\sqrt{(x-q_1)^2 + q_2^2} = x - q_1 + O(1/x)$, $(x \to \infty)$. This implies that $f(x) = p_1 - q_1 + O(1/x)$, $(x \to \infty)$. Hence, the limit of $d(v, F(v)) - d(v, N(v))$, where the point v goes to infinity on the edge e, exists. (If e is unbounded to the right as well, then we must have $p_1 = q_1$. In this case, it does not matter if x goes to $+\infty$ or $-\infty$. In both cases, the function f converges to zero.)

[1] We write $f(h) = O(g(h))$, $(h \to 0)$, if there are positive constants c and h_0 such that $|f(h)| \leq c \cdot |g(h)|$ for all $|h| \leq h_0$. Note that $f(h)$ may be negative.

We can prove that f converges monotonically by considering its derivative for large values of x. We have $f'(x) = (x - q_1)/\sqrt{(x-q_1)^2 + q_2^2} - (x - p_1)/\sqrt{(x-p_1)^2 + p_2^2}$. If $f'(x) = 0$, then $(x - q_1)^2 p_2^2 = (x - p_1)^2 q_2^2$. Since $p \neq q$, this equation has at most two solutions. Hence, for x large enough, the sign of f' does not change any more, which proves that the function f converges monotonically to the value $p_1 - q_1$. ∎

Theorem 1 follows from the following two lemmas.

Lemma 2 *Let e be an unbounded edge of $NVD(S)$ or $FVD(S)$. Then, $w(e) \geq width(S)$.*

Proof: Assume w.l.o.g. that e is horizontal and unbounded to the right. We consider the case that e is an edge of the closest-point Voronoi diagram of S. (The case that e is an edge of the furthest-point Voronoi diagram of S can be treated symmetrically.) Let $p = (p_1, p_2)$ be one of the two nearest neighbors of e. If we walk along e far enough to the right, then the furthest neighbor will not change any more. Let $q = (q_1, q_2)$ be this "final" furthest neighbor of e. By Lemma 1, $w(e)$ is equal to $p_1 - q_1$.

Let l (resp. l') be the vertical line through q (resp. p). We claim that all points of S are contained in the vertical slab bounded by l and l'. This will prove that the width of this slab, which is $w(e)$, is at least equal to the width of S.

Assume there is a point $r \in S$ that is to the left of l. Then, for all points x on e that are far enough to the right, we have $d(x, r) > d(x, q)$, i.e., q is not the furthest neighbor of x. This is a contradiction. Hence, all points of S are on or to the right of l. By a symmetric argument, it follows that all points of S are on or to the left of l'. ∎

Lemma 3
1. *There is an unbounded edge e of $NVD(S)$, such that $w(e) = width(S)$.*
2. *There is an unbounded edge e of $FVD(S)$, such that $w(e) = width(S)$.*

Proof: Let l and l' be two parallel lines that are at distance $width(S)$ such that all points of S are contained in the slab bounded by l and l'. It is known that each of l and l' contains at least one point of S, and at least one of l and l' contains at least two points of S. (See [6].)

Assume w.l.o.g. that l' contains two points of S. Also, assume w.l.o.g. that l and l' are vertical, and that l' coincides with l or is to the right of it.

Let p and p' be two points on l' such that there are no points of S between p and p' on l'. Let b be the perpendicular bisector of p and p'. Finally, let q be a point of S on l having maximal distance to b.

Any point x on b that is sufficiently far to the right has p and p' as its nearest neighbors and q as its furthest neighbor. Therefore, the part of b to the right of l' contains an unbounded edge of the closest-point Voronoi diagram of S. Call this unbounded edge e. It follows from Lemma 1 that $w(e)$ is equal to the distance between l and l', which is exactly the width of S. This proves the first claim. The second claim can be proved in a similar way by observing that the part of b to the left of l contains an unbounded edge of the furthest-point Voronoi diagram of S. ∎

The characterization of Theorem 1 leads to an (optimal) $O(n \log n)$ time algorithm for computing the width of a set S of n points. Since this algorithm is similar to that of Houle and Toussaint [6], we leave the details to the reader.

3 Characterizing the roundness of a planar point set

Let S be a set of points in the plane. If there is a point $x \in \mathbb{R}^2$ such that $d(x, F(x)) = d(x, N(x))$, then all points of S lie on a circle with center x, and the roundness of S is equal to zero. In this case, x is a vertex of $NVD(S)$ (and of $FVD(S)$) and, hence, Theorem 2 holds. Therefore, we make the following assumption.

Assumption 1 *For all points x in the plane, $d(x, F(x)) > d(x, N(x))$.*

3.1 It suffices to consider points on G

Lemma 4 *Let C be a circle and let x be the highest point of C. Let y and z be points on the boundary of C such that y is to the left of the vertical line through x and z is to the right of this line. Let C' be a circle such that x is on its boundary and y and z are both in its interior. Then the radius of C' is larger than the radius of C.*

Proof: W.l.o.g., let the center of C be the origin. Let l be the line through x and y. Let y' be the intersection of l with C' such that $y' \neq x$. Let b be the perpendicular bisector of x and y, and let b' be the perpendicular bisector of x and y'. Let m be the line through x and z. Let z' be the intersection of m with C' such that $z' \neq x$. Let c be the perpendicular bisector of x and z, and let c' be the perpendicular bisector of x and z'. Then (i) the center of C is the intersection of b and c, (ii) the center of C' is the intersection of b' and c', (iii) b' is parallel to and below b, and (iv) c' is parallel to and below c.

Hence, the center of C' is below both b and c. Since the angle made by b (resp. c) with the positive X-axis is in the interval $(\pi/2, \pi)$ (resp. $(0, \pi/2)$), the center of C' is below the center of C. This proves that the distance between x and the center of C' is larger than the distance between x and the center of C. Since x is on the boundary of both circles, this implies that the radius of C' is larger than that of C. ∎

Lemma 5 *We have* $rd(S) = \inf\{d(v, F(v)) - d(v, N(v)) : v \in G\}$, *where* $v \in G$ *means that* v *is a vertex of* G *or a point on an edge of* G.

Proof: Let x be a point of $\mathbb{R}^2$ that is in the interior of a face of G. That is, $x \notin G$. We will prove that there is a vertex v of G or a point v on an edge of G such that $d(v, F(v)) - d(v, N(v)) \leq d(x, F(x)) - d(x, N(x))$. This will prove the lemma.

Let p (resp. q) be the nearest (resp. furthest) neighbor of x. Assume w.l.o.g. that q is vertically below x. (That is, q is on the ray emanating downwards from x.)

Let C be the circle with center q that contains x on its boundary. Since p is the nearest neighbor of x, p is above the horizontal line through q.

Let f be the face of G that contains x. Start walking from x along the boundary of C in counterclockwise order, and let y be the point on the boundary of f that is encountered.

Why does y exist? Let x' be the lowest point of C. Then, q is closer to x' than p is. Hence, p is not the nearest neighbor of x'. As a result, while walking from x to x' along the left boundary of C, we must cross the boundary of the face f. This proves that y exists.

Similarly, let z be the point on the boundary of f that we encounter by walking along C in clockwise order, starting at x. Note that $y \in G$ and $z \in G$.

The claim is that $d(y, F(y)) - d(y, N(y)) \leq d(x, F(x)) - d(x, N(x))$, or $d(z, F(z)) - d(z, N(z)) \leq d(x, F(x)) - d(x, N(x))$.

Note that $N(x) = N(y) = N(z) = p$ and $F(x) = F(y) = F(z) = q$. To prove the claim, we assume that $d(y, q) - d(y, p) > d(x, q) - d(x, p)$ and $d(z, q) - d(z, p) > d(x, q) - d(x, p)$. Since $d(x, q) = d(y, q) = d(z, q)$, it follows that $d(x, p) > d(y, p)$ and $d(x, p) > d(z, p)$.

Let C' be the circle with center p that contains x on its boundary. Then, y and z are both contained in the interior of C'. Hence, by Lemma 4, the radius of C' is larger than that of C, i.e., $d(x, p) > d(x, q)$. This is a contradiction, because p is the nearest neighbor of x. ∎

3.2 Considering points in the interior of edges of G

Lemma 6 *Let* e *be any edge of* G, *and let* v *be a point in the interior of* e. *Then there is a point* w *on* e *such that* $d(w, F(w)) - d(w, N(w)) < d(v, F(v)) - d(v, N(v))$.

Proof: We can assume w.l.o.g. that e is contained in the X-axis, and that v is the origin.

If e is (part of) an edge of $NVD(S)$, then the interior of e has exactly two nearest neighbors. Similarly, if e is (part of) an edge of $FVD(S)$, then the interior of e has exactly two furthest neighbors. Note that e can be (part of) an edge of both $NVD(S)$ and $FVD(S)$, since edges of $NVD(S)$ and $FVD(S)$ can overlap.

Let $p = (p_1, p_2)$ (resp. $q = (q_1, q_2)$) be a nearest (resp. furthest) neighbor of e. We claim that p and q are not both contained in the X-axis. To prove this, assume that e is (part of) an edge of $NVD(S)$. Let $p' \neq p$ be the "other" nearest neighbor of e. Then e is contained in the perpendicular bisector of p and p', which is the X-axis. Hence, since $p \neq p'$, it follows that p does not lie on the X-axis. If e is (part of) an edge of $FVD(S)$, then it follows in a symmetric way that q does not lie on the X-axis. Of course, if e is both (part of) an $NVD(S)$-edge and an $FVD(S)$-edge, then both p and q do not lie on the X-axis.

Because v is not a vertex of G, there is a circle γ centered at v, having a positive radius, such that any point inside γ and on e has p as its nearest neighbor and q as its furthest neighbor.

Consider the function $f(x) := \sqrt{(x - q_1)^2 + q_2^2} - \sqrt{(x - p_1)^2 + p_2^2}$. Then for any real number x such that $|x|$ is small enough, the point $w := (x, 0)$ is contained in γ, and $f(x) = d(w, F(w)) - d(w, N(w))$. Hence, it suffices to show that there is a real number x, such that $|x|$ is small and $f(x) < f(0)$. We will prove this by considering the derivative of f for small values of x. Let $||q||$ denote the distance between q and the origin, i.e., $||q|| = \sqrt{q_1^2 + q_2^2}$. We have $f'(x) = (x - q_1)/\sqrt{(x - q_1)^2 + q_2^2} - (x - p_1)/\sqrt{(x - p_1)^2 + p_2^2}$. Since $\sqrt{(x - q_1)^2 + q_2^2} = ||q||(1 - xq_1/||q||^2 + O(x^2))$, $(x \to 0)$, we have for $x \to 0$,

$$\frac{x - q_1}{\sqrt{(x - q_1)^2 + q_2^2}} = \frac{x - q_1}{||q||(1 - xq_1/||q||^2 + O(x^2))}.$$

Using the asymptotic expansion $1/(1 - h) = 1 + h + O(h^2)$, $(h \to 0)$, where $h = xq_1/||q||^2 + O(x^2)$, we get

$$f'(x) = \frac{p_1}{||p||} - \frac{q_1}{||q||} + \left(\frac{1}{||q||} - \frac{1}{||p||} - \frac{q_1^2}{||q||^3} + \frac{p_1^2}{||p||^3}\right) x + O(x^2), (x \to 0).$$

We consider three cases.

Case 1: $p_1/||p|| > q_1/||q||$. Then, for all x such that $|x|$ is small enough, $f'(x) > 0$. Hence, for $x < 0$ and $|x|$ small enough, we have $f(x) < f(0)$.

Case 2: $p_1/||p|| < q_1/||q||$. Then, for $x > 0$ and $|x|$ small enough, we have $f(x) < f(0)$.

Case 3: $p_1/||p|| = q_1/||q||$. We know that p_2 and q_2 are not both equal to zero. This implies that $p_2 \neq 0$. We have

$$f'(x) = \left(\frac{1}{||p||} - \frac{1}{||q||}\right)\left(\frac{p_1^2}{||p||^2} - 1\right) x + O(x^2), (x \to 0). \quad (1)$$

By Assumption 1, we have $d(v, q) > d(v, p)$, i.e., $||q|| > ||p||$, implying that $1/||p|| - 1/||q|| > 0$. Also, $p_1^2 < p_1^2 +$

$p_2^2 = ||p||^2$, implying that $p_1^2/||p||^2 - 1 < 0$. Hence, the coefficient of x in (1) is negative. It follows that for $x > 0$ and $|x|$ small enough, we have $f(x) < f(0)$. This completes the proof of the lemma. ∎

3.3 The proof of the first part of Theorem 2

If all points of S lie on a line, then $rd(S) = width(S)$, and the first part of Theorem 2 holds. Therefore, we assume from now on that not all points of S lie on a line. This implies that G does not contain edges that are unbounded in two directions.

Recall that we assume edges of G to be closed. Hence, if e is a bounded edge, then its two endpoints, which are vertices of G, belong to e. Similarly, if e is an unbounded edge, then its one endpoint is a vertex of G, and it belongs to e.

For each unbounded edge e of G, we do the following. We cut e into two parts e_0 and e_∞, such that (i) e_0 is closed and bounded, and (ii) e_∞ is closed and unbounded, and, for $v \in e_\infty$, the function $d(v, F(v)) - d(v, N(v))$ is monotone if v moves along e_∞ to infinity.

Let E_b be the union of the set of all bounded edges of G, and the set of all edge parts e_0, where e ranges over the unbounded edges of G. Let E_i be the set consisting of all edge parts e_∞, where e ranges over all unbounded edges of G such that the function $d(v, F(v)) - d(v, N(v))$ is increasing if v moves along e_∞ to infinity. (Later, in Case 2 of the proof of Section 3.4, we will see that the set E_i need not be empty.) Finally, let E_d be the set consisting of all edge parts e_∞, where e ranges over all unbounded edges of G such that the function $d(v, F(v)) - d(v, N(v))$ is non-increasing if v moves along e_∞ to infinity.

Then, Lemma 5 implies that the roundness of S is equal to the minimum of

$$\inf\{d(v, F(v)) - d(v, N(v)) : v \in \text{edge of } E_b\}, \quad (2)$$

$$\inf\{d(v, F(v)) - d(v, N(v)) : v \in \text{edge of } E_i\}, \quad (3)$$

$$\inf\{d(v, F(v)) - d(v, N(v)) : v \in \text{edge of } E_d\}. \quad (4)$$

Since the function $d(v, F(v)) - d(v, N(v))$ is continous, and the set E_b is closed and bounded, we can replace (2) by

$$\min\{d(v, F(v)) - d(v, N(v)) : v \in \text{edge of } E_b\}. \quad (5)$$

Consider an edge part e_∞ of the set E_i. Let e be the corresponding edge of G, and consider the bounded part e_0 of e. It follows from the definition of E_i that

$$\inf\{d(v, F(v)) - d(v, N(v)) : v \in e_0\} \leq \inf\{d(v, F(v)) - d(v, N(v)) : v \in e_\infty\}.$$

Hence, the value of (3) is at least equal to that of (5).

Consider an edge part e_∞ of the set E_d. Let e be the corresponding edge of G. We have $\inf\{d(v, F(v)) - d(v, N(v)) : v \in e_\infty\} = w(e)$. Then, Lemma 2 implies that the value of (4) is at least equal to the width of S.

By Lemma 3, there is an unbounded edge e in G such that $w(e) = width(S)$. Then, the definition of roundness implies that $rd(S) \leq width(S)$.

Hence, at this moment, we know that the roundness of S is equal to the minimum of the values (4) and (5), that the value of (4) is at least equal to the width of S, and that the roundness of S is at most equal to the width of S. It follows that the roundness of S is equal to the minimum of (i) $\min\{d(v, F(v)) - d(v, N(v)) : v \in \text{edge of } E_b\}$, and (ii) the width of S.

Now we can complete the proof of the first part of Theorem 2. Let e be an edge or edge part of the set E_b.

First assume that e is a bounded edge of G. If v is a point in the interior of e, then we know from Lemma 6 that there is a point w on e such that $d(w, F(w)) - d(w, N(w)) < d(v, F(v)) - d(v, N(v))$. Hence, we can conclude from the definition of roundness that $rd(S) < d(v, F(v)) - d(v, N(v))$.

Next assume that e is part of an unbounded edge of G. Then, e has two endpoints, and exactly one of these is a vertex of G. Let p be this vertex. If v is a point on e such that $v \neq p$, then we know from Lemma 6 that there is a point w on e such that $d(w, F(w)) - d(w, N(w)) < d(v, F(v)) - d(v, N(v))$. Again, we can conclude that $rd(S) < d(v, F(v)) - d(v, N(v))$.

This proves that the roundness of S is equal to the minimum of

1. $\min\{d(v, F(v)) - d(v, N(v)) : v \text{ is a vertex of } G\}$, and

2. the width of S,

which is exactly the claim in the first part of Theorem 2.

3.4 The proof of the second part of Theorem 2

Suppose there is an optimal slab such that one of its bounding lines contains exactly one point of S. We will prove that $rd(S) < width(S)$.

Let l and l' be the bounding lines of an optimal slab. Assume that l contains exactly one point, say p, of S. Assume w.l.o.g. that l is the Y-axis, and that l' is to the right of l. It follows from [6] that we may assume that l' contains at least two points of S. There are two possible cases.

Case 1: l' does not contain a point of S whose Y-coordinate is equal to p's Y-coordinate.

Let p have coordinates (p_1, p_2). Let q be a point of S that lies on l', and that is closest to the line $Y = p_2$.

For any positive real number x, let C_x (resp. C'_x) be the circle with center (x, p_2) that contains p (resp. q) on its boundary.

If x is large enough (but still finite), then all points of S are inside or on the boundary of C_x: This is clearly true for p. Let r be any point of $S \setminus \{p\}$. Then r is to the right of l. Let r' be the intersection between the line $Y = p_2$ and the perpendicular bisector of p and r. If (x, p_2) is to the right of r', then r is contained in C_x. Let r be a point of $S \setminus \{p\}$ for which r' has maximal X-coordinate. Then for each point (x, p_2) to the right of r', all points of S are inside or on the boundary of C_x. In a similar way, it can be proved that there is a point r'' on the line $Y = p_2$ such that for each point (x, p_2) to the right of r'', all points of S are outside or on the boundary of C'_x.

Note that the radius of C_x is equal to x, and the radius of C'_x is strictly larger than the distance between (x, p_2) and the line l'. It follows that there is a finite-radius annulus containing all points of S that has width less than the distance between l and l'. This proves that $rd(S) < width(S)$.

Case 2: l' contains a point of S whose Y-coordinate is equal to p's Y-coordinate.

Let p have coordinates (p_1, p_2). Let q be the point of S on l' with Y-coordinate p_2, and let q have X-coordinate q_1. Let q' be another point of S on l' such that the line segment qq' does not contain points of S. We assume w.l.o.g. that q' is below q, and that the perpendicular bisector of q and q' is the X-axis. Hence, q' has coordinates $(q_1, -p_2)$.

For any real number $x \geq q_1$, let C_x (resp. C'_x) be the circle with center $(x, 0)$ that contains p (resp. q) on its boundary. In a similar way as in Case 1, it can be shown that if x is large enough (but still finite), then all points of S are inside or on the boundary of C_x. Also, if x is large enough (but still finite), then all points of S are outside or on the boundary of C'_x. The radius of C_x (resp. C'_x) is equal to $\sqrt{(x-p_1)^2 + p_2^2}$ (resp. $\sqrt{(x-q_1)^2 + p_2^2}$). We will prove that the difference of these radii is strictly less than $q_1 - p_1$ provided x is large enough. This will prove that there is a finite-radius annulus containing all points of S that has width less than the distance between l and l'. Hence, $rd(S) < width(S)$.

We analyze the asymptotic behavior of the radii for large values of x. Using elementary asymptotic methods, it follows that, for large x, the difference of the radii of C_x and C'_x is equal to

$$q_1 - p_1 + \frac{1}{2}(p_1 - q_1)p_2^2/x^2 + O(1/x^3), (x \to \infty).$$

Since $p_1 - q_1 < 0$, it follows that this difference is strictly less than $q_1 - p_1$ for x large enough. This completes the proof of the second part of Theorem 2.

3.5 An example

In Section 1, we already mentioned that for a set S of points on a line we have $rd(S) = width(S)$. We now give an example of a point set, not all on a line, for which $rd(S) = width(S)$ and any finite-radius annulus containing the set has width larger than $width(S)$. Note that by Theorem 2, each bounding line of each optimal slab must then contain at least two points.

Let $p = (0,0)$, $q = (0,2)$, $r = (1,1)$, $s = (1,3)$, $t = (1/2, -5)$, and $u = (1/2, 10)$. Let S be a set of points that contains all these six points, such that all other points are strictly between the lines $X = 0$ and $X = 1$.

Lemma 7 *Any finite-radius annulus containing all points of S has width larger than one.*

Lemma 8 $rd(S) = width(S) = 1$.

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