Consecutive guards[1]

M. Abellanas[2]
J. García-López[2]
F. Hurtado[3]

Abstract

In this paper we define consecutive vertex guards and consecutive edge guards in a simple polygon. We solve the corresponding Art Gallery problem obtaining the numbers $g_v(n)$ and $g_e(n)$ of vertex and edge guards always sufficient and sometimes necessary to guard a simple n-gon. We also obtain linear algorithms to place these guards, and solve the algorithmic problem of finding the minimun number of consecutive guards for a simple polygon. Finally we extend the last result to the problem of finding the minimum length path on the boundary that guards the polygon.

1 Introduction

The classical results in Art Gallery [1-7] solve problems of this kind: *How many guards are always sufficient and sometimes necessary to guard a simple n-gon?* The reason why the question is this and not: *Given a simple n-gon what is the minimum number of guards needed to guard it?*, is that the answer for the latter question depends on the polygon. So the second one is actualy an algorithmic question. Adding the fact that the algorithmic problem is NP-complete [8] it is clear that the most interesting problem is the first. But this situation will change for consecutive guards.

A set of vertex guards in a polygon will be called a set of consecutive guards if they are placed in consecutive vertices of the polygon. In a same way we define a set of consecutive edge guards.

For this kind of guards we solve the Art Gallery problem, obtaining linear algorithms to put them and proving that the problem of the minimum number of guards it is not NP. We obtain algorithms with complexity $O(n^2\log(n))$ for the last problem in both cases: vertex and edge guards.

Using the same ideas that in the two last algorithms we obtain an other algorithm. with the same complexity, that computes the minimum length path on the boundary of the polygon from where the polygon is visible or is guarded.

Let us introduce some notation that will be used in this paper. P will denote a simple polygon with n vertices and it will be described by the counterclockwise sorted list of its vertices $p_1p_2p_3\dots p_n$. The list of edges will be denoted by $e_1e_2e_3\dots e_n$ where the edge e_i is the segment with endpoints p_i and p_{i+1} for all $i \in \{1,\dots,n\}$ asuming that p_{n+1} denotes p_1.

[1] **Partilly supported by CICYT grant TIC 93-0747-C02**

[2] **Universidad Politécnica de Madrid.**

[3] **Universidad Politécnica de Cataluña.**

2 Consecutive guards

Let us denote by $g_v(n)$ the number of consecutive vertex guards always sufficient and sometimes necessary to guard a simple n-gon. In order to prove **theorem 1** we introduce a previous lemma that allows us to make a shorter proof of the theorem.

Lemma 1

a) A polygon with a guard in every vertex except two is guarded.

b) A polygon with a concave vertex p_i that has a guard in every vertex except p_i, p_{i-1} and p_{i+1} is guarded.

Proof: Consider a triangulation of the polygon. Then every triangle has at least one vertex where there is a guard because the unique triangle that could have the three vertices without guards is $p_{i-1}p_ip_{i+1}$ (only for the case *b*). But this triangle can not belong to the triangulation because p_i is a concave vertex. So every triangle is guarded and then the polygon.

$\triangle$

Theorem 1 $g_v(n)$ is defined as follows:

$$g_v(n) = \begin{cases} 1 & \text{if} \quad n = 3 \\ n-3 & \text{if} \quad n \geq 4 \text{ and even} \\ n-4 & \text{if} \quad n \geq 5 \text{ and odd} \end{cases}$$

Proof: A- $g_v(n)$ guards are sometimes necessary:

i) If $n = 3$ or $n = 4$ the result is obvious.

ii) If $n \geq 6$ and even the "windmill" with $\frac{n}{2}$ tips (figure 1) needs $n-3$ guards.

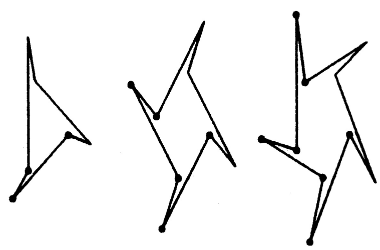

Figure 1

iii) If $n \geq 5$ and odd the "comb" with $\frac{n-1}{2}$ tips (figure 2) needs $n-4$ guards.

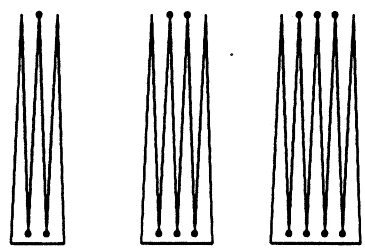

Figure 2

B- $g_v(n)$ guards are always sufficient:

i) If $n = 3$ or $n = 4$ the result is true because these polygons are star-shaped.

ii) If $n \geq 6$ and even. Suppose that the polygon is not convex, in the other case the result is obvious. Then there is a concave vertex p_i and by **lemma 1** putting guards in every vertex except p_{i-1}, p_i and p_{i+1} the polygon is guarded.

iii) If $n \geq 5$ and odd. Consider a triangulation of a the polygon and his dual tree T. First we prove that there exists a node of T with degree 2.

Suppose by contradiction that every node of T has degree 1 or 3 and denote by k_1 and k_2 the number of nodes in each case. It is easy to prove by induction that $k_1 = k_2 + 2$, so the number of nodes $2k_2 + 2$ is even and this means that n is also even. This contradicts the initial hypothesis.

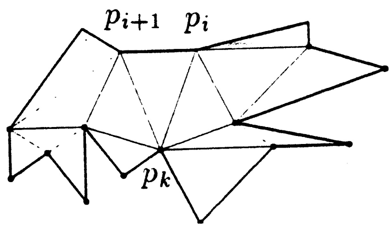

Figure 3

Let $p_k p_i p_{i+1}$ be one triangle which corresponds to a 2 degree node of T (see figure 3). Note that $k \neq i,\ i+1$ because the degree of the node is 2. Putting guards in every vertex except p_{i-1}, p_i, p_{i+1} and p_{i+2} the polygon is guarded. To see that we split the polygon by cutting the diagonals $p_k p_i$ and $p_k p_{i+1}$. The three remaining subpolygons are guarded by **lemma 1**.

△

For edge guards we denote by $g_e(n)$ the number of consecutive edge guards always sufficient and sometimes necessary to guard a simple n-gon.

In this paper we only prove that $g_e(n)$ guards are sometimes necessary, the complete proof can be seen in the full paper version. This part needs a very simple arithmetic property shown in the following lemma.

Lemma 2 The equation $5n_1 + 3n_2 = k$ has integer solution $n_1, n_2 \geq 0$ for every integer $k \geq 8$.

Proof. The result is true for $k = 8, 9$ and 10. For any $k > 10$ we decompose k in this way: $k = 3 * k_1 + k_2$ where $k_1 > 0$ and $k_2 = 8, 9$ or 10. Then we express $k_2 = 5n'_1 + 3n'_2$ and finally $k = 5n'_1 + 3(n'_2 + k_1)$.

△

Theorem 2 $g_e(n)$ is defined as follows:

$$g_e(n) = \begin{cases} 1 & \text{if} \quad 3 \leq n \leq 5 \\ n-5 & \text{if} \quad n \geq 6 \end{cases}$$

Proof. A- $g_e(n)$ guards are sometimes necessary:

i) If $3 \leq n \leq 6$ the result is obvious.

ii) If $n = 7$ figure 4 shows a 7-gon that needs 2 edge guards.

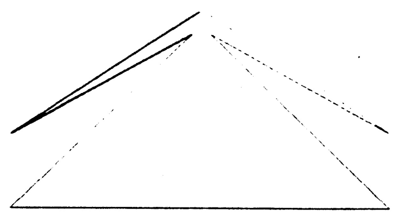

Figure 4

iii) If $n \geq 8$ we decompose $n = 5n_1 + n_2$ (such decomposition exists by **lemma 2**). The polygon with n_1 arrows and n_2 tips (figure 5) needs $n - 5$ guards.

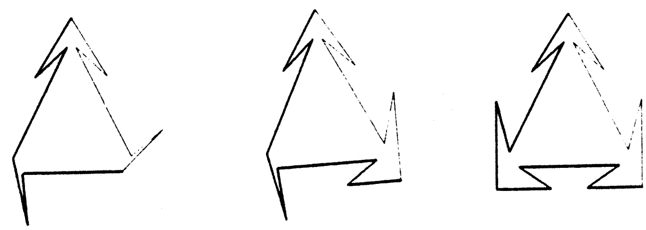

Figure 5

B- $g_e(n)$ guards are always sufficient: See the full paper for this.

$\triangle$

3 Placing guards

Given a simple n-gon we want to put a number of consecutive guards less or equal than $g_v(n)$ such that the polygon was guarded. We do that with the **Algorithm 1** following the ideas of the first theorem proof. This means that we need a triangulation of a polygon.

Algorithm 1

Input: A simple polygon P and one triangulation T of it.

Step 1: Compute the number n of vertices.

Step 2: If n is even:

- If the polygon is convex put one guard in one vertex. End.
- Else find a concave vertex p_i and put guards in every vertex except p_{i-1}, p_i and p_{i+1}. End.

Step 3: If n is odd find a triangle $p_k p_i p_{i+1}$ $(k \neq i, i+1)$ whose node in T has degree 2. Put guards in every vertex except p_{i-1}, p_i, p_{i+1} and p_{i+2}. End.

All steps have linear complexity and then the **algorithm 1** is linear. We want to note that **theorem 1** can be proved without using any triangulation of a polygon and this proof provides a linear algorithm whose input is just a simple polygon. We also remark that if the input is an even size polygon then there is an algorithm that runs in constant time. The proof of **theorem 2** allows a linear algorithm to place $g_e(n)$ consecutive edge guards in a simple n-gon. All these results are included in the full paper version.

4 Minimum number of guards

It is known that a set of vertex guards that sees the boundary of a simple polygon does not necessary guard the polygon (figure 6.a). But for consecutive vertex guards this result is true, and for consecutive edge guards it is sufficient to see all vertices in order to guard the polygon. This fact is proved in the following lemma.

Lemma 3

a) A set of consecutive vertex guards that sees the boundary of a simple polygon guards the polygon.

b) A set of consecutive edge guards that sees all vertices of a simple polygon guards the polygon.

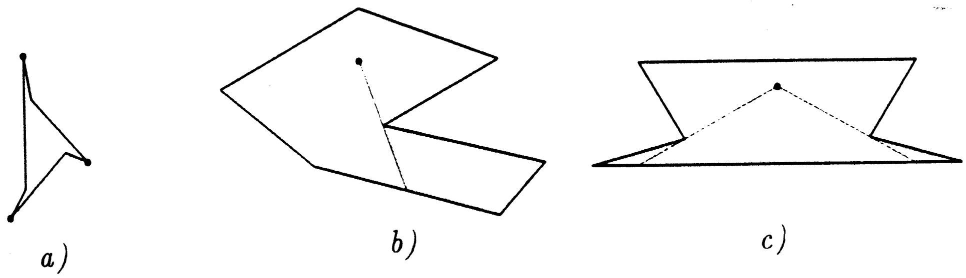

Figure 6

Proof: Suppose by contradiction that there exists a point p which is not seen by any guard. Then all guards are placed in one pocket (or two consecutive pockets for the case a) of the visibility polygon fo p. (see figure 6.b and 6.c). In case a we conclude that there exists a point on the boundary that can not be seen by any guard and in case b one vertex that can not be seen by any guard. This contradicts our hypothesis and concludes the proof.

△

We split the boundary of the polygon P in segments such that every point in each of them sees the same set of vertices (figure 7). This can be done computing the vertices of all visibility polygons and sorting them counterclockwise on the boundary. We denote by L the sorted list of them, define an interval like a set of consecutive segments of L and denote by $L_1, L_2 \ldots L_n$ the lists of intervals that describe the visibility polygons from the vertices. The complexity of these is $O(n)$ and for L is $O(n^2)$.

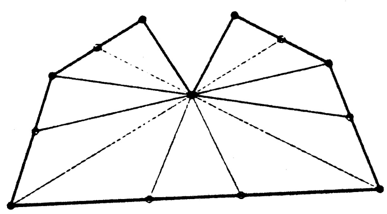

Figure 7

The idea for the algorithm is to compute the minimum set S_i of consecutive vertex guards starting in each vertex p_i and going counterclockwise. Finally we choose the minimum set of guards between the n computed set of guards. If S_i has been computed, S_{i+1} will be obtained by removing the list $L_{\text{first}(S_i)}$ and by adding the list $L_{\text{last}(S_i)+1}$ (where last(S_i) and first(S_i) denotes respectively the last and first vertices in the list S_i), until the polygon has been completely guarded.

Each segment in L has a level that indicates the number of vertices, actualy in the set of guards, that see the segment. Then, adding (removing) the list L_i increases (decreases) by 1 the level of the visible segments from p_i.

Algorithm 2

Input: A simple polygon P.

Step 1: Compute all visibility polygons from vertices and the lists $L, L_1, L_2, \ldots L_n$.

Step 2: Set to 0 the level of each segment in L, $S = \emptyset$ and $i = 1$.

Step 3: Add($L_{\mathbf{last}(S)+1}$), $S = S + p_{\mathbf{last}(S)+1}$.

Step 4: If S does not guard the polygon go to step 3.

Step 5: $S_i = S$, $i = i + 1$, Remove($L_{\mathbf{first}(S)}$), $S = S - p_{\mathbf{first}(S)}$.

Step 6: If $i < n$ go to Step 3.

Step 7: Choose the minimum set of $S_1 \ldots S_n$. End.

Implementing the list L in a balanced binary tree with a field for the level, the functions add and remove need $O(n\log(n))$ time. Function remove is called $n-1$ times and function add at most $2n-1$ times. Then the cost of steps 3,4 and 5 is $O(n^2\log(n))$. It is easy to see that the rest of steps do not increase the complexity.

This algorithm can be fitted with the same complexity to solve the problem for edge guards and for the minimum length path over the boundary that guards the polygon. See the full paper for this. The main open question is how to improve the complexity of these algorithms.

5 References

[1] J. O'Rourke, Art Gallery Theorems and Algorithms, *Oxford University Press* (1987).

[2] V. Chvátal, A combinatorial theorem in plane geometry, *J. Combin. Theory Ser. B* 18 (1975).

[3] S. Fisk, A short proof of Chavátal's watchman theorem, *J. Combin. Theory Ser. B* 24 (1978).

[4] F. Hoffmann, "On a rectilinear art gallery problem, *Proc. Int. Colloq. Automata. Languages and Programming 90*, LNCS 443, 1990, pp. 717-728.

[5] T. C. Shermer, Recent results in art galleries, *Proceedings of the IEEE*, vol 80, No. 9, September 1992.

[6] F. Hoffmann, M. Kaufmann and K. Kriegel, The art gallery theorems for polygons with holes, *Proc. Foundations of Computer Science 91* pp. 39-48.

[7] G.F. Jennings and W. J. Lenhart, An art gallery theorem for line segments in the plane, *Pattern Recognition Letters* Special Issue on Computational Geometry, G.T. Toussaint, Ed., September 1993.

[8] D.T. Lee and A.K. Lin, Computational complexity of art gallery problems, *IEEE Trans. Info. Th* IT-32 (1986b) 276-282.