Controlling Guards

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Extended Abstract

1 Introduction

The original Art Gallery Problem raised by V. Klee asks how many guards are sufficient to watch every point inside a n-sided simple polygon. A point x is visible to a guard y if the line segment $\overline{xy}$ does not intersect the exterior of the polygon. Chvátal showed that $\lfloor \frac{n}{3} \rfloor$ guards are always sufficient and sometimes necessary to watch any polygon of n edges [1]. Many variations of this theorem have been explored and many results have been obtained, [4,6]. In 1987, Shermer introduces the concept of hidden guard set [7]. A hidden guard set is a guard set of points such that no two guards in the set are visible to each other. This corresponds to "independent dominating sets" in $PVG(P)$, the point visibility graph of P. Shermer showed that not every polygon admits a hidden guard set on its vertices and that, given a polygon P, the problem of determining whether such a hidden guard set exists for P is NP-complete.

In this paper we analyze another variation on art gallery problems: in contrast to hide guards we require that every guard must be watched by another guard. We define $VG(S,P)$, the visibility graph of a set of guards S in a polygon P as follows: the vertex set is S and there is an edge between two guards if they are visible to each other in P. we consider two kinds of guards: **connected guards**, such that $VG(S,P)$ is connected and **watched guards**, such that each one is watched, at least, by another guard, i.e., in $VG(S,P)$ there are not isolated vertices. In this paper, following [2], we analyze the combinatorial aspects of these kinds of guards. In [3], Liaw, Huang and Lee deal the algorithmic aspects of connected guards which call cooperative guards. They showed that the *minimum cooperative guards problem, MCG problem*, (given a polygon P, find the minimum number of cooperative guards necessary to cover P) is NP-hard. Also they presented linear algorithms for solving the problem on 1-spiral and 2-spiral polygons.

For connected guards we prove that, $g^W(n)$, the minimum number of watched guards necessary to watch any polygon of n vertices, $n \geq 5$, is $\lfloor \frac{2n}{5} \rfloor$.

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If we consider connected guards, then $g^C(n)$, the minimum number of connected guards, is $\lfloor \frac{n}{2} \rfloor - 1$, being the bound $\frac{n}{2} - 2$ when the polygon is an orthogonal polygon. The lower bounds are given by means of examples in section 2. The proofs of the upper bounds are given in sections 3 and 4.

2 The lower bounds

For watched guards the lower bound is $\lfloor \frac{2n}{5} \rfloor$. It is clear that a polygon with 5, 6 or 7 vertices needs 2 watched guards. In figure 1 we show an octogon that requires 3 watched guards.

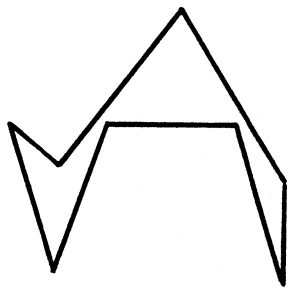

Figure 1. A polygon with 8 edges, $t = 6$, requiring 3 watched guards

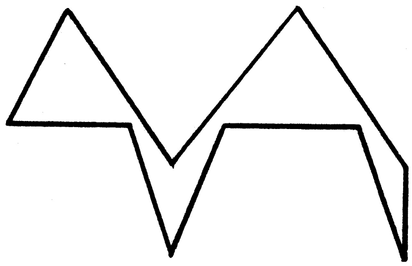

Figure 2. A polygon with 10 edges, $t = 8$, requiring 4 watched guards

A polygon with 10 vertices that needs 4 watched guards is shown in figure 2. This polygon can be generalized to a polygon with $t = 5k + 3$ triangles needing $2k + 2$ watched guards (see figure 3).

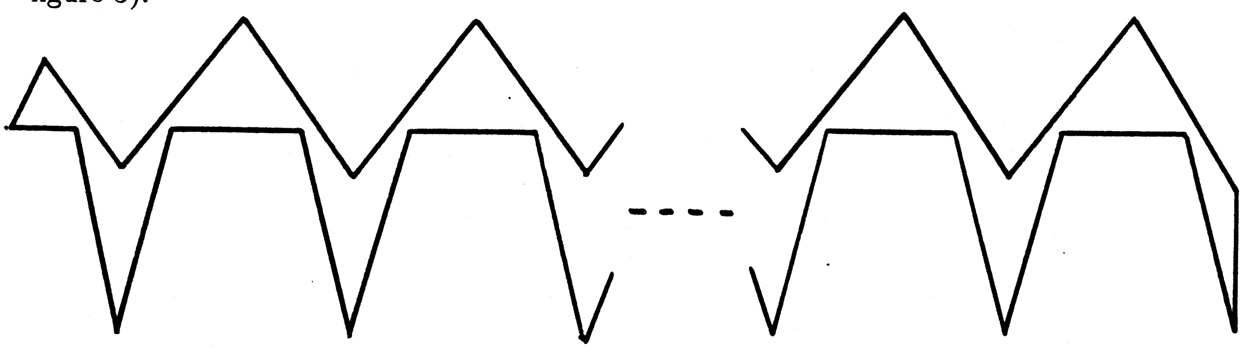

Figure 3. A polygon with $5k + 3$ triangles requiring $2k + 2$ watched guards

Therefore, if P is a polygon with n vertices and t is the number of triangles in one triangulation of P, $t = n - 2$

$$g^W(n) \geq \lceil \frac{2t}{5} \rceil = \lceil \frac{2(n-2)}{5} \rceil = \lfloor \frac{2n}{5} \rfloor$$

This establishes the lower bound for watched guards.

If we consider connected guards, the snake polygon shown in figure 4 requires $\lfloor \frac{n}{2} \rfloor - 1$ connected guards.

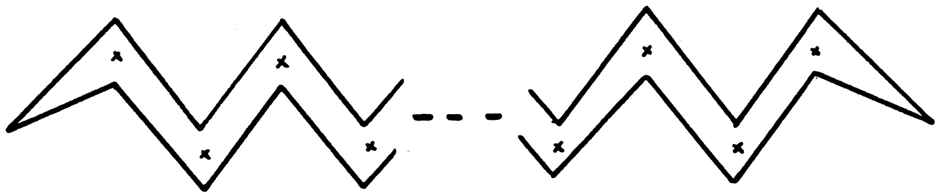

Figure 4. The "snake" polygon needs $\lfloor \frac{n}{2} \rfloor - 1$ connected guards

For orthogonal polygons the lower bound is $\frac{n}{2} - 2$. Looking at the most simple periodic staircase (fig. 5), we check that $\frac{n}{2} - 2$ connected guards are necessary and sufficient to watch this polygon. Thus we establish the lower bounds for connected guards.

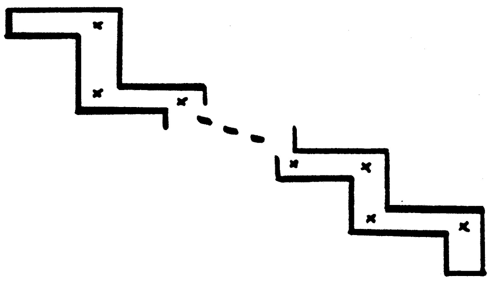

Figure 5. An orthogonal polygon that needs $\frac{n}{2} - 2$ connected guards

3 Upper bound for watched guards

The upper bound for watched guards is $\lfloor \frac{2n}{5} \rfloor$. The proof is by induction and it follows the main outlines of O'Rourke's proof for mobile guards [5]. Let P a polygon and T a triangulation graph for P. O'Rourke uses the identity between the number of combinatorial and geometric mobile guards necessary and sufficient to dominate and cover triangulation graphs and polygons, respectively.

Now the combinatorial counterpart of the watched guards are the vertex guards in T such that any two of them are linked by an arc of T. It is clear that if a triangulation graph of a polygon can be dominated by k combinatorial guards, then the polygon can be covered, i. e. watched, by k geometric watched guards placed at vertices. This implies that a proof of the sufficiency of $\lfloor \frac{2n}{5} \rfloor$ combinatorial watched guards in a triangulation graph establishes the sufficiency of the same number of geometric watched guards in a polygon with $n > 4$ vertices.

Following O'Rourke's proof we must consider the edge contraction of T and utilize the following lemmas:

Lemma 1(O'Rourke)
Let T a triangulation graph of a polygon P, and T' the graph resulting from an edge contraction of T. Then T' is a triangulation graph of some polygon P'.

Lemma 2
Suppose that $f(n)$ combinatorial watched guards are always sufficient to dominate any n-node triangulation graph. Then if T is an arbitrary triangulation graph of a polygon P with one vertex guard, Q, placed at any one of its n nodes, then an additional $f(n-1)$ watched guards are sufficient to dominate T. (But, perhaps the guard Q remains without any guard watching to it).

Then we establish the sufficiency of the bound $\lfloor \frac{2n}{5} \rfloor$ for small triangulation graphs and the existence of a diagonal that will allow us to get the induction step.

Lemma 3
(a) Every triangulation graph of a pentagon can be dominated by two combinatorial watched guards with one at any selected node.
(b) Every triangulation graph of a hexagon can be dominated by two combinatorial watched guards with one at one vertex of any selected edge.
(c) Every triangulation graph of a n-polygon with $7 \geq n \geq 11$ can be dominated by $\lfloor \frac{2n}{5} \rfloor$.

Lemma 4
If T is any triangulation graph of a polygon P, with $n \geq 12$ vertices then there exists a diagonal that cuts off exactly $6, 7, 8, 9$ or 10 edges.

With the preceding lemmas available, the induction proof is a simple enumeration of cases.

Theorem 1
Every triangulation graph T of a polygon with $n \geq 5$ vertices can be dominated by $\lfloor \frac{2n}{5} \rfloor$ combinatorial watched guards.

Proof.
Lemma 3 establishes the truth of the theorem for $5 \leq n \leq 11$. Assume now, that $n \geq 12$ and the induction hypothesis. Lemma 4 establishes that there is a diagonal that partitions T into two graphs T_1 and T_2, where T_1 contains k boundary edges with $6 \leq k \leq 10$. We must consider each value of k separately. Here we include only one case.

Case $k = 9$. (See figure 6)
The presence of any of the diagonals $(0,8),(0,7),(0,6),(1,9),(2,9),(3,9)$ would violate the minimality of k. So that the triangle L in T_1 that is bounded by d is either $(0,4,9)$ or $(0,5,9)$, which are equivalent cases. Suppose L is $(0,5,9)$. Form the graph T_0 by adjoining the polygon 056789 to T_2. The polygon 012345 has six edges, and so by Lemma 3(b), it can be dominated with two combinatorial watched guards, one of them placed at node 0 or 5. We choose node 5. T_0 has $n-4$ edges and the guard at node 5 permits the remainder of T_0 to be dominated by $f(n-4-1) = f(n-5)$ combinatorial watched guards, where $f(m)$ is the number of combinatorial watched guards that are always sufficient to dominate a triangulating graph of m nodes. By the induction hypothesis, $f(m) = \lfloor \frac{2m}{5} \rfloor$.

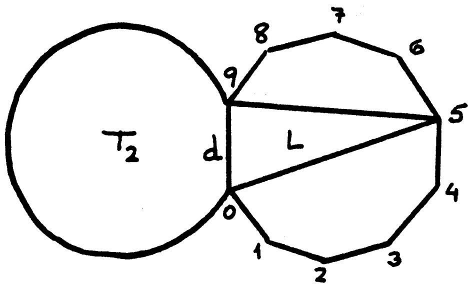

Figure 6. Case $k = 9$

Therefore, $\lfloor \frac{2(n-5)}{5} \rfloor = \lfloor \frac{2n}{5} \rfloor - 2$ combinatorial watched guards suffice to dominate T_0. Together with the two allocated in the polygon 012345, we conclude that T is dominated by $\lfloor \frac{2n}{5} \rfloor$ combinatorial watched guards.

4 Upper bounds for connected guards

The upper bound for connected guards in general polygons is $\lfloor \frac{n}{2} \rfloor$ -1. The proof is similar to that for watched guards but more simple. Here we omit the proof of this

Theorem 2
The minimum number of connected guards necessary to watch any polygon of n vertices is

$$g^C(n) = \lfloor \frac{n-2}{2} \rfloor$$

In the orthogonal case the upper bound is $\frac{n}{2} - 2$. Let P an orthogonal polygon with n vertices. We consider a convex quadrilateralization of the polygon P and let q the number of quadrilaterals Let be S the set of guards built placing a guard at every diagonal of P that share two convex quadrilaterals. It is easy to check that $VG(S,P)$ is connected and $card(S) = q - 1 = \frac{n}{2} - 2$. Therefore, $\frac{n}{2} - 2$ connected guards are always sufficient to watch any orthogonal polygon of n vertices. Thus we have shown the following

Theorem 3
The minimum number of connected guards necessary to watch any orthogonal polygon of n vertices is

$$g_O^C(n) = \frac{n}{2} - 2$$

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