K-Guarding Polygons on The Plane

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Abstract

A polygon is called k-guardable if it is possible to find a collection of points Q *in the interior of the edges of* P such that every point in P is visible from at least k elements in Q *and no edge of* P *contains more than one element in* Q. In this paper we prove that every simple polygon can be 2-guarded with at most n-1 guards. We prove that any simple polygon with n vertices can be 1-guarded using at most $\lfloor \frac{n-1}{2} \rfloor$ guards. We also prove that not every polygon with holes is 2-guardable but that they are always 1-guardable. Our proofs lead to linear time algorithms to find 1- and 2-guarding collections for simple polygons.

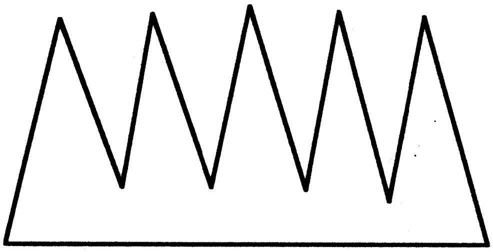
1. Introduction

How many guards are necessary, and how many are sufficient to patrol an art gallery—especially a modern one, with its numerous alcoves, corners, and narrow snake-like passages? Fueled by the modern interest in combinatorial and computational geometry, this apparently naïve question of combinatorial geometry has, since its formulation [V. Klee (1973), cf. R. Honsberger (1976)], stimulated a rush of papers, surveys, and even a book [J. O'Rourke (1987)], most written in the last decade.

The mathematical beginnings are found in the well-known gem of a theorem by V. Chvátal (1975) according to which $\lfloor \frac{n}{3} \rfloor$ stationary guards are occasionally necessary and always sufficient. A *guard* or simply, a *light* is a stationary light source which can survey 360° about its fixed, designated position. In this formulation the gallery, having straight walls, is a polygon of n vertices and, for every point of the polygon and its interior, there is a light (guard) which illuminates it.

There are numerous other interesting variations of art gallery problems, for instance, traditional orthogonal art galleries all of whose walls are either horizontal or vertical, in which case $\lfloor \frac{n}{4} \rfloor$ guards are necessary and sufficient [KKK83]; mobile guards, each of whom patrols from along a line segment within an n-vertex polygon, of whom $\lfloor \frac{n}{4} \rfloor$ are necessary and sufficient [O'Ro83]. In [CRUZ] it is shown that a rectangular art gallery with n exposition rooms (that is one housed in a rectangular building subdivided into n rectangular rooms) can always be guarded with exactly $\lfloor \frac{n}{2} \rfloor$ guards. In a different direction, Fejes Toth [FeTo77] has proved that any family of n disjoint closed convex sets on the plane can be illuminated with at most 4n-7 lamps. Recently, variations to guarding problems in which restrictions on the angle of illumination, or visibility of the guards are imposed have been introduced. In these *floodlight illumination problems*, the assumption that a guard can see 360° about its fixed, designated position is no longer valid [BGLOSU93, CRU93].

In this context we study the following variation to guarding problems proposed by A. Lubiw at the open problem session of the Fourth Canadian Conference in Computational Geometry: Let P be a simple polygon with n vertices. We say that P is k-guardable if it is possible to find a set of points Q consisting of *interior points of edges of* P such that every point of P is visible from at least k elements in Q and *no edge of* P *has more than one element in* Q. For what values of k is every simple polygon k-guardable? It has been observed by T. Shermer [She92] that *comb* polygons [Chv75, O'R87] are not 3-guardable; such a polygon is shown in Figure 1.



A polygon which is not 3-guardable

Figure 1

In this paper we prove that every simple polygon with n vertices can be 2-guarded using at most n-1 points. We also prove that any simple polygon with n vertices can be 1-guarded with at most $\lfloor \frac{n}{2} \rfloor$ guards. These bounds are tight up to an additive constant. We prove that any polygon with one hole is also 2-guardable. We also prove that every polygon with holes is 1-guardable, and that it is not true that every polygon with holes is 2-guardable.

2. One and Two-Guarding Simple Polygons

In this section, we consider the problem of 1-guarding and 2-guarding simple polygons. To facilitate our presentation, we will assume that a is not contained in the line joining any two vertices of P and that for every edge e of P the line containing e contains no vertex of P other than the end vertices of e. This condition may be easily dropped, leaving our result unchanged. We proceed now to prove our first result:

Theorem 1: Every simple polygon can be two-guarded with at most n-1 guards.
Proof: Let a be any point on the interior of an edge of P and let P_a be the *visibility polygon of* a, that is the set of all points $q \in P$ such that the line joining a with q is contained in P. Notice that P_a may contain vertices that are not vertices of P and that some edges of P may have up to two vertices of P_a in their interior. (See Figure 2.) Let v be a vertex of P_a that is not a vertex of P. The line joining v to a contains a vertex of P, which we shall denote by v_a. Let e be an edge of P that has two vertices of P_a in its interior, say b and c. Notice that b and c_a are mutually visible in P_a (the triangle formed by a, b and c is contained in P_a). Thus the line segment joining them is contained in P_a. Remove from P_a the triangle determined by b, c and c_a. Apply this procedure to all edges of P containing two vertices of P_a that are not vertices of P and name the resulting polygon P^1_a. Place a guard at all the vertices of P^1_a that are not vertices of P_a. If an edge e of P is completely visible from a place one guard in the middle of it and finally place one guard at the point a itself. (See Figure 2).

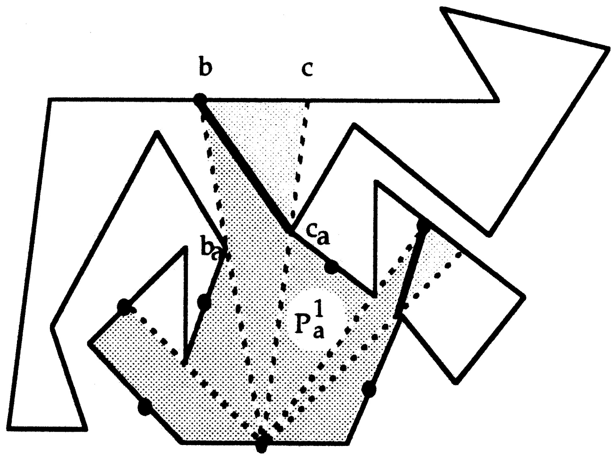


Figure 2

Observation 1: All points in P^1_a are 2-guarded (by a and at least one of the other guards placed on the boundary of P^1_a).

Clearly $P-P^1_a$ can be "broken" into several simple polygons $P_1,...,P_k$ with disjoint interiors with the property that each one of them contains exactly one vertex that is not a vertex of P. We will denote such a vertex by v(i), i=1,...,k. Notice that some pairs of elements of $P_1,...,P_k$ may have at most one point in common, i.e. a vertex of P^1_a that is not a vertex of P. Now we process each P_i using the following recursive procedure:

Procedure 2-Guarding (P_i, v(i))

Calculate the visibility polygon $P_{v(i)}$ of v(i) in P_i. Two cases arise:

a) $P_i = P_{v(i)}$. In this case place a guard in the middle of each edge of $P_{v(i)}$ except for the two edges of $P_{v(i)}$ containing v(i).

b) $P_{v(i)} \neq P_i$. Three cases are considered now:

i) An edge e of P_i completely visible from v(i). Place a guard in the middle of e.

ii) An edge of P_i containing exactly one vertex v of $P_{v(i)}$ that is not a vertex of P_i. Place a guard at v.

iii) For each edge e of P_i containing two vertices of $P_{v(i)}$, say b and c, that are not vertices of P_i proceed as follows: Locate the reflex vertices $b_{a(i)}$ and $c_{v(i)}$ of P_i contained in the interior of the line segment joining v(i) to b and c respectively. Join $b_{a(i)}$ to c with a line segment and delete from $P_{v(i)}$ the triangle with vertices b, c and $b_{a(i)}$. Place a guard at c. Let $P^1_{v(i)}$ be the polygon obtained from P_i after deleting all the triangles generated by edges containing two vertices of $P_{v(i)}$ not vertices of $P_{v(i)}$. Partition $P_{v(i)}-P^1_{v(i)}$ into m simple polygons $P_1,\ldots,P_m$ each containing exactly one vertex v(j) that is not a vertex of P_i, j=1,..., m. For j=1,..., m execute **2-Guarding** (P_j, v(j)).

End 2-Guarding

It now follows by Observation 1 that the collection of guards thus obtained is a 2-guarding of P, that is each visibility subpolygon $P_{v(i)}$ calculated during our execution of **2-Guarding** is 2-guarded. Moreover, our procedure places at most one guard on each edge of P. In Figure 3 we present the 2-guarding produced for the polygon in Figure 2.

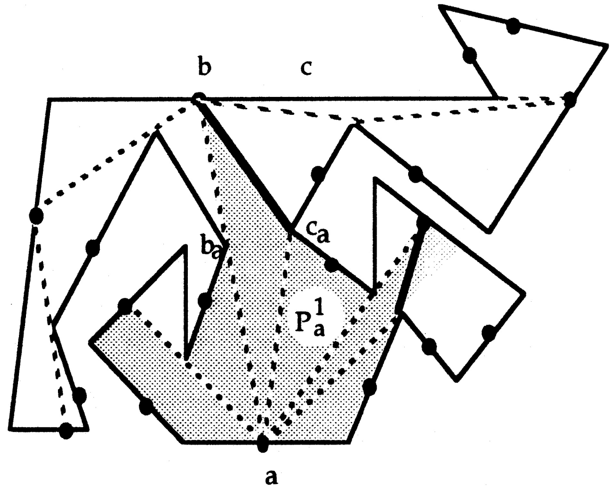
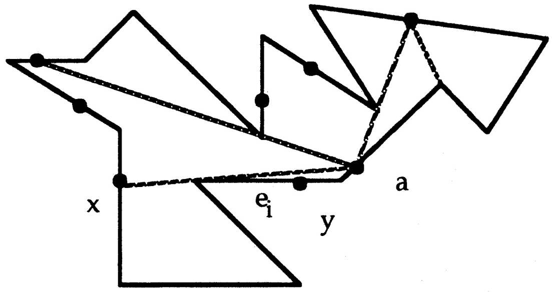
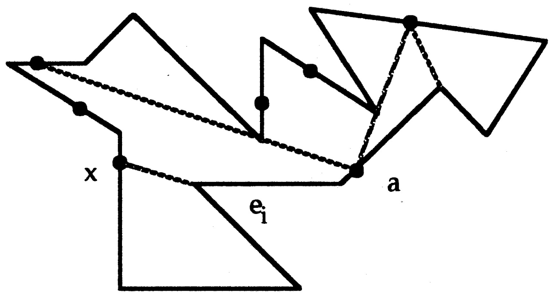


Figure 3



Original Guarding



Modified Guarding

Figure 4

We now show that in the above procedure, we can save one guard by careful choice of the initial guard a. To this end, let v_i be a convex vertex of P incident to edges e_{i-1} and e_i. If we place a on e_{i-1} close enough to v_i so that a can see all of e_{i-1} then a guard x is created by the ray generated by a and v_{i+1} and a guard y is placed on e_i. (See Figure 4.) If we now move x up until it is above the line generated by e_i until it sees a, then we can eliminate the guard originally placed on e_i.

(See Figure 4.) This proves that n-1 guards suffice to 2-guard P.

QED

Theorem 2: $\lfloor \frac{n}{2} \rfloor$ guards are always sufficient and sometimes necessary to 1-guard a simple polygon.

Proof: In the proof of Theorem 1, color the initial point a with color 1 and the guards generated by a with color 2. In the successive iterations, if a guard was generated by a guard with color 1 (resp. 2), color it with color 2 (resp. 1). By observation 1, and our coloring rule, it follows that every point is seen by at least one point with color 1 and one with color 2. Choose the color class with fewer vertices to obtain the sufficiency of our result. The family of comb polygons similar to the polygon shown in Figure 1 demonstrates that $\lfloor \frac{n}{2} \rfloor$ guards are sometimes required.

QED

3. Polygons With Holes

Given a simple polygon P', and k disjoint polygons $Q_1,\ldots,Q_k$ contained in the interior of P, we say that the polygon $P=P'-(Q_1 \cup \ldots \cup Q_k)$ is a polygon with k holes.

In this section we study the problem of 1- and 2-guarding for polygons with holes. We start by proving:

Theorem 3: Not every polygon with holes is 2-guardable.

Proof: To prove Theorem 2, all we have to do is to exibit a polygon with two holes that is not 2-guardable. To this end consider the polygon with two holes shown in Figure 5.

Consider the point set S={a,b,c,d,e,f}. In order to two-guard the elements of S, we can choose only guards placed in the interior of $e_1,\ldots,e_{11}$. Moreover, no guard placed in any of these edges can see two elements of S. Our result now follows.

QED

Next we prove:

Theorem 4: Every polygon with holes is 1-guardable.

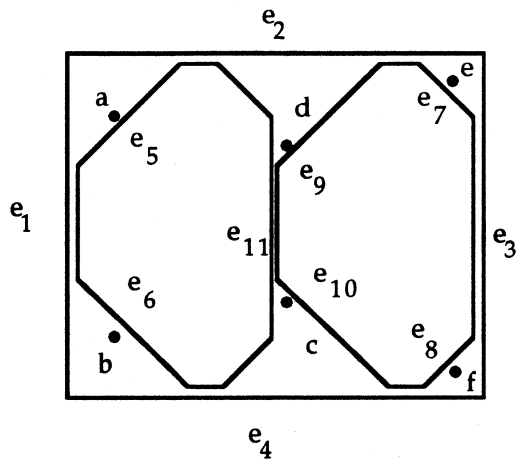


Figure 5

Before we proceed with the proof of Theorem 3 we recall the following result on visibility.

Lemma 2: Let $S=\{\lambda_1,\ldots,\lambda_n\}$ be a collection of disjoint line segments and p a point on the plane such that p is externally visible from S, i.e. there is a ray starting at p that does not intersect any element of S. Then S contains at least one line segment λ_i that is completely visible from p.

A proof of this lemma can be obtained from results presented in [FRU]. It is easy to see that p induces an order relation "<" in S as follows:

i) We say that λ_a blocks λ_b (denoted by $\lambda_a \rightarrow \lambda_b$) if there is a point q in λ_b such that the line segment joining p to q intersects λ_a.

ii) We now say that $\lambda_a < \lambda_b$ if $\lambda_a \rightarrow \lambda_b$ or there is a chain of elements $\lambda_a = \lambda_1 \rightarrow \lambda_2 \ldots \rightarrow \lambda_k = \lambda_b$.

In the language of [FRU] "<" is a light source order. Thus the element λ_i claimed in Lemma 2 is nothing else than a minimal element of the order relation "<" on S.

Proof of Theorem 4: Let P be a polygon with holes. Without loss of generality, assume that no edge of P is parallel to the x-axis, that no two vertices of P have the same y-coordinate, and that the difference between the ycoordinates of any two such vertices is at least $\varepsilon > 0$.

For every vertex v of P consider the longest line segment contained in P which is parallel to the x-axis and contains v. These lines partition P into a collection of convex polygons $T=\{R_1,....,R_m\}$ with disjoint interiors. For every edge e of P place a guard in its interior at distance at most $\frac{\varepsilon}{2}$ from its lower end point.

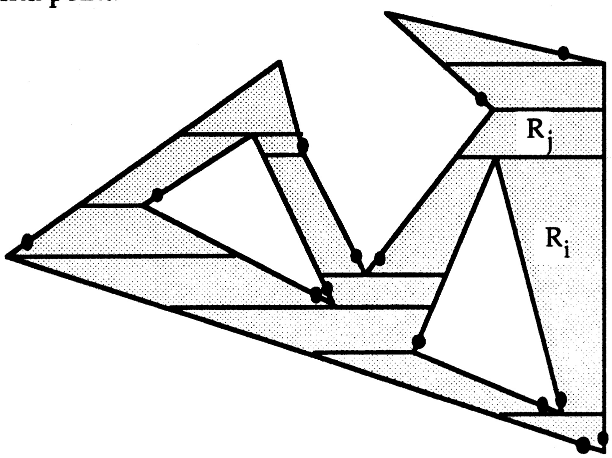


Figure 6

We claim that these points 1-guard P. In order to prove our claim we observe that if the boundary of a region R_i of T intersects the interior of an edge e and also contains its lower end-point, then it contains the guard assigned to e. Suppose then that an element R_j of T does not contain a guard in its boundary and consider a point p in R_j. If p lies in a line segment contained in P that contains an edge e of P, then the guard assigned to e guards p. Suppose then that this is not the case. Using the horizontal line through p, cut the polygon P in two parts and delete that part of P above it. (See Figure 6.) At all the remaining vertices of P, cut away a sufficiently small segment from each edge of P, or the remaining segment of an edge of P. (See Figure 6).

Notice that we get a disjoint family of line segments for which p is externally visible. By Lemma 2, one of these segments, say e', is completely visible from p. Since p is in the interior of P, it follows that p sees the side of e' facing towards the interior of P, and thus the guard assigned to the edge of P that contains e' guards p.

QED

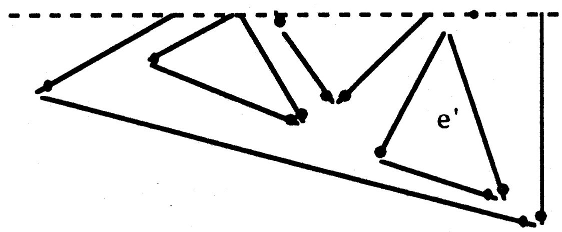


Figure 7

4. Conclusions

In this paper we studied the problem of guarding a polygon P by using at most one guard in the interior of each edge of P. We proved that any polygon can be 2-guarded using at most n-1 guards. We also proved that any polygon can be 1-guarded with $\lfloor\frac{n-1}{2}\rfloor$. We proved that not all polygons with holes are 2-guardable and that they are always 1-guardable. An open problem here is that of deciding if a polygon with holes is 2-guardable. As for 1- and 2-guarding of simple polygons, it is easy to see that it is possible to develop linear time algorithms to find 1- and 2-guardings of polygons using n-1 guards and $\lfloor\frac{n-1}{2}\rfloor$ guards respectively. Details of this will be given in the full version of this paper.

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