# All Convex Polyhedra can be Clamped with Parallel Jaw Grippers *

Prosenjit Bose David Bremner Godfried Toussaint

School of Computer Science
McGill University

## Abstract

We study various classes of polyhedra that can be clamped using parallel jaw grippers. We show that all $n$-vertex convex polyhedra can be clamped regardless of the gripper size and present an $O(n+k)$ time algorithm to compute all positions of a polyhedron that allow a valid clamp where $k$ is the number of antipodal pairs of features. We also show that all terrain polyhedra and rectilinear polyhedra can be clamped and a valid clamp can be found in linear time.

## 1 Introduction

Grasping is a well known research area in robotics. Much research has been done on the problem of gripping or immobilizing an object with a multifingered hand [3, 7, 10, 11, 13]. Recently, researchers have considered the problem of finding a "geometric" grip of a planar object [14, 6, 1, 15]. Souvaine and Van Wyk [14] studied the problem of clamping a polygon with a pair of parallel line segments, motivated by robot hands known as *parallel jaw grippers* that are pairs of parallel plates. Each plate is referred to as a *gripper*. Informally, a polygon $P$ is *clamped* in the plane when it is "securely" held between the two grippers (modeled in the plane by a pair of line segments forming the opposite sides of a rectangle) such that $P$ does not rotate or slip out of the gripper when the gripper is squeezed. A polygon is called *clampable* if there exists a clamp for every positive length gripper.

Souvaine and Van Wyk [14] showed that all convex polygons are clampable, and conjectured that all simple polygons are clampable. Capoyleas [6] gave a slightly weaker definition of clamping and showed that polygons whose pockets are convex terrains with respect to the lid are clampable under this definition. Albertson, Haas, and O'Rourke [1] defined the class of *free* polygons (a polygon is called free if no outward normal intersects the interior) and showed that free polygons, sail polygons, and polygons with at most 5 vertices are clampable.

In this paper we address the problem of determining when a 3-dimensional object (modeled as a simple polyhedron) is clampable with a parallel jaw gripper consisting of a pair of parallel plates. First, we observe that terrain polyhedra and rectilinear polyhedra are clampable and that a valid clamp can be found in linear time. The main result of our paper is that all convex polyhedra are clampable. We give an $O(n+k)$ time algorithm to find all clamps of an $n$-vertex convex polyhedron, where $k$ is the number of antipodal pairs of features. Some proofs and details are omitted in this extended abstract. For full proofs, we refer the reader to the technical report [4].

## 2 Notation and Preliminaries

Let us first introduce some of the terminology we will be using in this paper.

A *polyhedron* in $E^3$ is a solid whose surface consists of a number of polygonal faces. A polyhedron is *simple* if its surface can be deformed continuously into the surface of a sphere. As we are only dealing with simple

*This research was supported in part by NSERC postgraduate scholarships, and grants NSERC-OGP0009293, FCAR-93ER0291. Address: 3480 University Street, Montréal, Québec, Canada, H3A 2A7. E-mail: jit@cs.mcgill.ca, bremner@cs.mcgill.ca, godfried@cs.mcgill.ca.

polyhedra, we will refer to them as polyhedra in the remainder of the paper. We will represent a simple polyhedron by a finite set of plane polygons such that every edge of a polygon is shared by exactly one other polygon (referred to as an adjacent polygon) and no subset of the polygons has the same property. The vertices and the edges of the polygons are the *vertices* and *edges* of the polyhedron. The polygons are the *facets* of the polyhedron.

A polyhedron partitions the space into two disjoint domains, the *interior* (bounded) and the *exterior* (unbounded). We will denote the open interior of the polyhedron $P$ by $int(P)$, the boundary by $bd(P)$, and the open exterior by $ext(P)$. The boundary is considered part of the polyhedron; that is, $P = int(P) \cup bd(P)$.

## 2.1 Grippers and Clamps

We give a geometric model of a *parallel jaw gripper* and a geometric definition of a *clamp.*

A parallel jaw gripper is modeled by a pair of rectangles forming opposite faces of a rectangular box. Each rectangle is referred to as a gripper. The size of a gripper is determined by the length and the width of the rectangle. Although intuitively it may seem that the size of a gripper plays a key role in determining whether or not a polyhedron can be clamped, we will show that several classes of polyhedra can be clamped regardless of the size of the gripper.

To give a geometric definition of a clamp, we begin by first defining the *contact set* (similarly to [14]).

**Definition 1** *Given that both grippers touch a polyhedron $P$ in some configuration, the* contact set *of the configuration is the set of all points $p$ on either gripper such that for all $\epsilon > 0$, there is a point $q$ that lies in both the interior of the box defined by the grippers and in the interior of $P$ such that $dist(p, q) < \epsilon$. (Thus, the contact set may be a proper subset of the set of all points at which the grippers touch polygon $P$). If each gripper contains at least one point in the contact set, then the grippers form a* grip *on $P$.*

The $\epsilon$-condition for the contact set ensures that the polyhedron is between the two grippers and avoids configurations such as the one in Figure 1. Although when a polyhedron is in a grip both grippers touch the polyhedron and at least some part of the interior of the polyhedron is contained between them, the grip is not necessarily secure. By this we mean that by squeezing the grippers, the polyhedron may rotate or slip out. Therefore, we wish to define a grip that "holds a polyhedron securely". Such a grip will be referred to as a *clamp.*

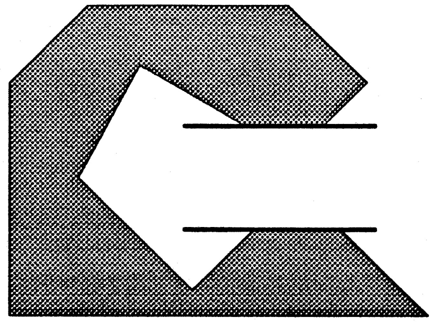

Figure 1: An empty contact set.

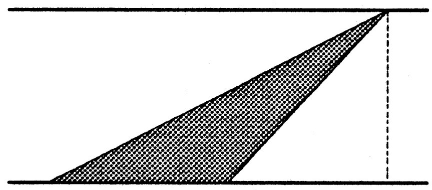

Figure 2: Unstable grip.

For the purposes of discussion, consider one of the grippers as a fixed horizontal table and the other gripper as a horizontal plate pushing down on the object. The object is thus immobilized if no point in the upper contact set moves under downward pressure. Suppose that the object is gripped as shown in Figure 2. Clearly, such a grip is unstable and the object will rotate. However, suppose that the object is a cube gripped by two opposing faces. The object is then in static equilibrium and thus immobilized. We present a geometric definition of a clamp that captures the difference between these two examples. A discussion of the mechanics behind this model can be found in [4].

**Definition 2** *A grip on a polyhedron $P$ is a* clamp *when one of the following two conditions holds:*

1. *one gripper $g_1$ contains a point $b$ in its contact*

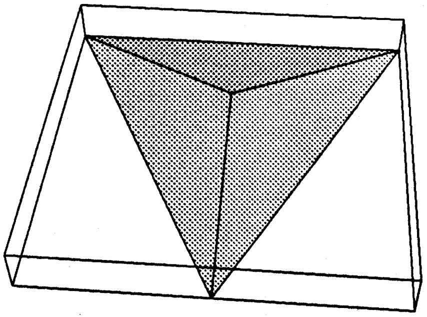

Figure 3: Only a vertex-face clamp exists.

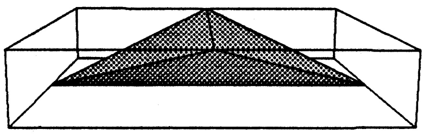

Figure 4: Only an edge-edge clamp exists.

*set and the other gripper $g_2$ contains three non-collinear points $c, d, e$ in its contact set such that the orthogonal projection of $b$ onto $g_2$ falls in the open triangle$(c, d, e)$.*

2. *one gripper $g_1$ contains a pair of points $a, b$ in its contact set and the other gripper $g_2$ contains a pair of points $c, d$ in its contact set such that the orthogonal projection of the open line segment $\overline{ab}$ onto $g_2$ properly intersects the open line segment $\overline{cd}$.*

The tetrahedron in Figure 3 can be clamped under condition 1 with one gripper on the peak of the tetrahedron and the other on its base. The tetrahedron in Figure 4 can be clamped under condition 2. Notice that the definition of a clamp does not include the grip shown in Figure 5 since both upper contact points project onto a line segment between two of the lower contact points. A modification of condition 1 such as — gripper $g_1$ contains two points $a, b$ in its contact set such that there is a point $x$ on the open line segment $\overline{ab}$ whose orthogonal projection onto $g_2$ falls in the open triangle$(c, d, e)$ — would have included this grip but since the objects we consider in this paper do not need such a grip, we chose to use the simpler definition. For a discussion of a more general definition of a clamp, we refer the reader to [4].

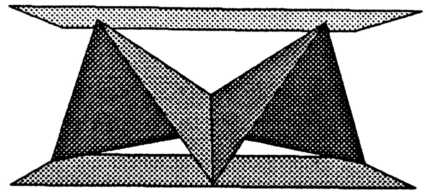

Figure 5: A grip that is not a clamp.

We say that a polyhedron is *clampable* if it admits a clamp for every size of gripper. We say that a polyhedron is *partially clampable* if it admits a clamp with a gripper of a particular size.

## 3 Clampable Polyhedra

The definition of a clamp immediately implies that all terrain polyhedra are clampable. A polyhedron $P$ is a terrain polyhedron provided that there exists a face $f \in P$ such that $\forall x \in P$ the open line segment $\overline{xy} \in int(P)$ where $y$ is the orthogonal projection of $x$ onto the plane containing $f$. Such polyhedra can be recognized in linear time [2].

**Observation 1** *Every terrain polyhedron can be clamped and a clamp can be determined in $O(n)$ time where $n$ is the number of vertices of the polyhedron.*

Terrain polygons can also be recognized in linear time [2] and under the definition of a clamp given in [14], they are clampable.

**Observation 2** *Every terrain polygon can be clamped and a clamp can be determined in $O(n)$ time where $n$ is the number of vertices of the polygon.*

Every rectilinear polygon has at least 4 extreme edges (edges incident on the minimum enclosing rectangle). Since the two edges adjacent to an extreme edge are parallel and project orthogonally onto each other for some positive distance, there exist at least 2 distinct clamps of a rectilinear polygon. Similarly, every rectilinear polyhedron has at least 6 extreme faces. By considering the set of extreme faces in a given direction, the problem reduces to finding a clamp on a set of rectilinear polygons; but this just requires finding an extreme edge of the set of polygons.

**Observation 3** *All rectilinear polyhedra and polygons are clampable and a valid clamp can be determined in $O(n)$ time.*

## 3.1 Convex polyhedra

In this subsection, we establish the main result of this paper: that all convex polyhedra may be clamped. We will also give an algorithm to determine all positions that admit a valid clamp.

To show that all convex polyhedra may be clamped, we must show that given a gripper of any size, there always exists at least one position of the grippers that satisfies one of the two conditions defining a clamp. A key step towards showing this is to notice that a convex polyhedron can only be gripped at an antipodal pair of features. Let us define what is meant by an antipodal pair of features.

A plane supporting a convex polyhedron is a plane that is tangent to the polyhedron with the polyhedron lying completely in one-half space of the plane. The set of tangent points will be referred to as the *supporting set.* The supporting sets of a pair of parallel supporting planes of a convex polyhedron form an *antipodal pair.* Since a plane of support can only meet a convex polyhedron at a vertex, edge or face, there can only be six types of antipodal pairs: vertex-vertex, vertex-edge, vertex-face, edge-edge, edge-face and face-face.

From the definition of a clamp, we see that a vertex-vertex pair and vertex-edge pair cannot form a clamp. Therefore, what remains to be shown is that there always exists an antipodal pair satisfying one of the two criteria of clamping. There are two special types of antipodal pairs that immediately come to mind: the diameter and the width. Since the diameter can be determined by a pair of vertices (see [12]), we can rule it out. The width, however, is characterized as follows.

**Lemma 1** *[9] The width of a convex polyhedron $P$ in three dimensions is the minimum distance between parallel planes of support passing through either an antipodal vertex-face pair or an antipodal edge-edge pair of $P$.*

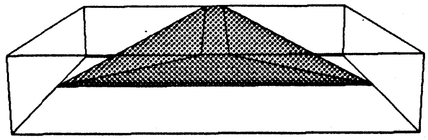

Figure 6: Only a face-edge clamp exists.

As stated, the lemma is somewhat misleading with respect to clamping. The width can also be determined by a face-face pair or an edge-face pair, but since both of these contain a vertex-face pair, they can be disregarded when considering the width. When viewing them in terms of a clamp, these other two pairs do play an important role. The proof of Lemma 1 in [9] shows that the width cannot be determined by a vertex-vertex pair or a vertex-edge pair. Therefore, we re-state the lemma as such.

**Lemma 2** *The width of a convex polyhedron $P$ in three dimensions is the minimum distance between parallel planes of support passing through either an antipodal vertex-face pair, edge-edge pair, face-face pair, or edge-face pair of $P$.*

Lemma 2 in itself is not sufficient to show that all convex polyhedra are clampable. We now show, however, that the width always satisfies one of the two conditions defining a clamp. Although there can be many positions admitting a valid clamp, there are some polyhedra (such as the ones depicted in Figures 3, 4, and 6) where the width is the only position defining a clamp. Before proving this theorem, we need to establish a few geometric lemmas.

Given two parallel planes $P_1$ and $P_2$, we denote the minimum distance between them by $D_{\min}(P_1, P_2)$.

**Lemma 3** *Let $P_1$ and $P_2$ be a pair of parallel planes with a point $x \in P_1$ and a point $y \in P_2$ such that line segment $\overline{xy}$ is orthogonal to both $P_1$ and $P_2$. For every other pair of parallel planes $Q_1$ and $Q_2$ with $Q_1$ containing $x$ and $Q_2$ containing $y$, we have that $D_{\min}(Q_1, Q_2) < D_{\min}(P_1, P_2)$.*

**Proof:** Let $z$ be the orthogonal projection of $x$ on $Q_2$. Notice that the length of $\overline{xz}$ is $D_{\min}(Q_1, Q_2)$ and the length of $\overline{xy}$ is $D_{\min}(P_1, P_2)$. However, triangle$(x, y, z)$

is a right triangle, with $\overline{xy}$ as hypotenuse. ∎

For the following lemma, we assume that the supporting plane is the $X - Y$ plane through the origin and that the convex polyhedron lies above this plane (i.e. all $z$ coordinates are non-negative). Let $L$ be an oriented line on this plane and $o$ a point on $L$. We define clockwise and counter-clockwise rotation of this plane as viewed by an observer at $o$ looking in the direction of $L$.

**Lemma 4** *Let $C$ be a convex polyhedron, and $P$ a supporting plane. For every oriented line $L$ in $P$ such that no point of the supporting set lies to the left of $L$, the plane $P$ can be rotated clockwise about $L$ by some $\epsilon > 0$ without intersecting $C$.*

We now have the tools to prove the following theorem.

**Theorem 1** *The width of a convex polyhedron determines a clamp for any positive size gripper.*

**Proof:** (sketch) Suppose the width is determined by an antipodal vertex-face pair that is not an edge-face or face-face pair. Orient the polyhedron such that the plane $P$ containing the face $f$ is the $X - Y$ plane and the polyhedron is above it. Let $P'$ be the plane parallel to $P$ containing the vertex $v$. If the width is not a clamp, then the vertex $v$ does not project into the interior of the face $f$. Let $x$ be the projection of $v$ onto $P$. Let $L$ be an oriented line in $P$ containing $x$ such that $f$ is to the right of $L$. By Lemma 4, both $P$ and $P'$ can be rotated by some $\epsilon$, such that they form another pair of parallel planes of support. However, by Lemma 3, this contradicts the fact that we had the width.

A similar argument holds for the other cases. ∎

We now turn our attention to computing a valid clamp. Theorem 1 guarantees that every convex polyhedron has at least one valid clamp. To find such a clamp, we rely on Brown's transformation [5]. Brown's technique allows one to efficiently compute all antipodal pairs of features.

Let us briefly summarize this transformation. Given an $n$-vertex convex polyhedron tangent to the $X - Y$ plane with no vertical faces (such an orientation can always be found), the first step is to divide the faces into an *upper* set and a *lower* set. The outward normals of the faces in the upper set have a positive $z$-component and the outward normals in the lower set have a negative $z$-component. This division has the property that any antipodal pair of features must have one plane of support tangent to the upper set and one plane of support tangent to the lower set.

Notice that the slope of a nonvertical plane is really a two-dimensional vector, so every such plane maps into a point in $\mathbb{R}^2$.

$$\text{Plane: } z = ax + by + c \rightarrow (a, b) \in \mathbb{R}^2$$

Under this point-plane transformation, a face of the polyhedron maps to a point since only one plane of support contains a face. Similarly, an edge $e$ of the polyhedron maps to an edge $e'$ where each point of $e'$ corresponds to the slope of a plane supporting edge $e$. Finally, a vertex of the polyhedron maps to a planar region, where each point of the planar region corresponds to the slope of a plane supporting the vertex. Brown outlines a method for computing in linear time the upper and lower subdivisions corresponding to the point-plane transformation of the upper and lower set, respectively.

Consider the convex subdivision overlay of the upper and lower subdivisions. A vertex $f$ of one subdivision that lies in a face $v$ of the other corresponds to a vertex-face antipodal pair of features because $f$ corresponds to a face and $v$ corresponds to a vertex. Similarly, two vertices that coincide correspond to a face-face antipodal pair. Also, two edges that properly intersect correspond to an edge-edge antipodal pair. Finally, an edge of one subdivision containing a vertex of the other corresponds to an edge-face antipodal pair.

To generate all valid clamps, we first generate all vertex-face, edge-face, edge-edge and face-face antipodal pairs. Once these pairs are computed, we need to verify that they satisfy the definition of a clamp. Theorem 1 guarantees that at least one pair will be valid.

We give a brief outline of the algorithm, which takes a convex polyhedron $P$ as input.

**Algorithm 1: Find all valid clamps.**

1. Transform $P$ into two planar subdivisions in $O(n)$ time as described in [5].
2. Compute the overlay of the two planar subdivisions using Guibas and Seidel's algorithm [8] in $O(n+k)$ time where $k$ is the size of the overlay.
3. Examine all vertices and all parallel rays of unbounded regions of the overlay to generate all vertex-face, edge-face, edge-edge and face-face pairs in $O(k)$ time.
4. Examine all pairs to determine their validity as a clamp in $O(k)$ time.
5. Output the valid clamps.

Verifying whether an edge-edge pair is a valid clamp can be done in constant time simply by determining whether the projection of one edge onto the other properly intersects it. On the other hand, verifying any one of the other three types of antipodal pairs for validity cannot be done in constant time. However, we are able to amortize the cost over all pairs of antipodal features to achieve $O(k)$ time. For example, the amount of time we take to check a vertex-face pair can be charged to the number of vertex-vertex pairs in that pair. A similar argument holds for the other two types of pairs. Therefore, we have the following theorem.

**Theorem 2** *Every n-vertex convex polyhedron is clampable and all clamping positions can be computed in $O(n+k)$ time where $k$ is the number of antipodal pairs of features.*

## References


[1] Albertson, M., R. Haas, J. O'Rourke, *Some results on clamping a polygon*, Tech. Rep. No. 003, Department of Computer Science, Smith College, 1990.

[2] Asberg, B., G. Blanco, P. Bose, J. Garcia, M. Overmars, G. Toussaint, G. Wilfong, and B. Zhu, Feasibility of design in stereolithography. *Proc. 13th Symp. on FST & TCS*, Bombay, India, pp. 228–237, 1993.

[3] Baker, B., S. Fortune, and E. Grosse, Stable prehension with a multi-fingered hand: extended abstract, *Proc. IEEE Intl. Conf. on Robotics and Automation*, St. Louis, Missouri, 1985.

[4] Bose, P., D. Bremner, and G. Toussaint, *All convex polyhedra can be clamped with parallel jaw grippers*, Technical Report, School of Computer Science, McGill University, to appear.

[5] Brown, K., *Geometric Transforms for Fast Geometric Algorithms*, Ph.D. Thesis, Department of Computer Science, Carnegie-Mellon University, Pittsburgh, Pa., 1979.

[6] Capoyleas, V., Clamping of polygonal objects. *Pattern Recognition Letters*, **14**, pp. 704–714, 1993.

[7] Czyzowicz, C., I. Stojmenovic and J. Urrutia, Immobilizing a shape in the plane, *Proc. 3rd Canad. Conf. Comput. Geom.*, Vancouver, B.C., pp. 41–45, 1991.

[8] Guibas, L., and R. Seidel, Computing convolutions by reciprocal search, *Discrete Comput. Geom.*, **2**, pp. 175–193, 1987.

[9] Houle, M., and G. Toussaint, Computing the width of a set. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, **10**, pp. 761–765, 1988.

[10] Mishra, B., J. Schwartz, and M. Sharir, On the existence and synthesis of multifinger positive grips, *Algorithmica*, **2**, pp. 541–588, 1987.

[11] Montejano, L., and J. Urrutia, Immobilizing figures on the plane, *Proc. 3rd Canad. Conf. Comput. Geom.*, Vancouver, B.C., pp. 46–49, 1991.

[12] Preparata, F.P., and M.I. Shamos, *Computational geometry – an introduction*. Springer-Verlag, New York, 1985.

[13] Salisbury, J., Design and control of an articulated hand, *Proc. 1st Intl. Symp. on Design and Synthesis*, Tokyo, Japan, 1984.

[14] Souvaine, D., and C. Van Wyk, *Clamping a polygon*. Tech. Rep. No. LCSR-TR-98, Department of Computer Science, Rutgers University, 1989.

[15] Teichmann, M., and B. Mishra, Reactive algorithms for grasping using a modified parallel jaw gripper, to appear in *Proc. IEEE Intl. Conf. on Robotics and Automation*. May 1994.