Triangulating with High Connectivity

Tamal Krishna Dey* Michael B. Dillencourt† Subir K. Ghosh‡

ABSTRACT

We consider the problem of triangulating a given point set, using straight-line edges, so that the resulting graph is "highly connected." Since the resulting graph is planar, it can be at most 5-connected. Under the nondegeneracy assumption that no three points are collinear, we characterize the point sets with three vertices on the convex hull that admit 4-connected triangulations. More generally, we characterize the planar point sets that admit triangulations having neither chords nor non-complex (i.e., nonfacial) triangles.

1 Introduction

We consider the problem of obtaining a planar network of maximum connectivity when the vertex locations are specified and only straight-line edges are allowed. Given a finite planar point set S and an integer k, we say that S is *k-connectible* if there exists a k-connected planar graph with straight-line edges having vertex-set S.

Planar point sets that are 1-, 2-, and 3- connectible are easily characterized, and no planar point set is k-connectible for $k > 5$. Our main result is a characterization of the conditions under which a planar point set in general position and having exactly 3 extreme vertices is 4-connectible (Theorem 3.2).[1] This is actually a consequence of a more general theorem, Theorem 3.1, which characterizes the conditions under which a planar point set in general position can form the vertices of a triangulation having neither chords nor complex (i.e., nonfacial) triangles. One consequence of these results is that any planar point set in general position becomes 4-connectible if we are allowed to add 2 additional (Steiner) points (Theorem 3.6). All our proofs are constructive, and the graphs can be constructed in $O(n \log n)$ time. In Section 4, we conclude and state several open problems. We omit several proofs and include only sketches of others; details can be found in the full paper.

We know of no previous work on the problem of determining whether a given set is k-connectible. In a sense, this problem is the inverse of the problem of drawing a planar graph, which has been the subject of considerable attention [2]. In particular, our results complement recent work, motivated by floorplanning problems in VLSI circuit design, concerning layout of triangulations having no complex triangles. A floorplan in VLSI circuit design is essentially a dissection of a rectangle into a finite number of non-overlapping sub-rectangles. It is known that a triangulated planar graph has a rectangular dual which is a floorplan only if it does not have a complex triangle [5]. Triangulations without complex triangles have been previously studied from a purely graph-theoretical perspective by one of the authors [3].

*Department of Computer Science, Indiana-Purdue University at Indianapolis, Indianapolis, IN 46202, USA. Permanent address: Department of Computer Science and Engineering, I.I.T., Kharagpur, India 721302.

†Department of Information and Computer Science, University of California, Irvine, California 92727, USA. The support of a UCI Faculty Research Grant is gratefully acknowledged.

‡Computer Science Group, Tata Institute for Fundamental Research, Bombay 400005, India.

[1] By general position, we mean that no three points are collinear. Most of the terms used in this section are defined in Section 2.

2 Preliminaries

Let S be a finite set of planar points. A *triangulation* of S is a planar graph T with vertex-set S such that all edges are line segments, the boundary of the outer face is the boundary of the convex hull, and all faces of T with the possible exception of the exterior face are bounded by triangles. A *chord* of a triangulation T is an edge connecting two nonconsecutive vertices on the boundary, and a *complex triangle* is a triangle that does not form the boundary of a face; see Figure 2.1. A triangulation is said to be *noncomplex* if it has neither chords nor complex triangles.

A graph is *k-connected* if it remains connected whenever $k-1$ vertices and their attached edges are removed. A planar point set S is *k-connectible* if there exists a k-connected planar graph with vertex set S such that all edges are line segments. Since adding edges to a graph cannot decrease the connectivity, S is k-connectible iff there is a k-connected triangulation with vertex set S.

The following characterizations of k-connected triangulations are immediate consequences of results established in [4].

Lemma 2.1 A triangulation is 3-connected if and only if it does not have a chord.

Lemma 2.2 A triangulation T is 4-connected if and only if

(A1) T does not have a chord.

(A2) T does not have a complex triangle.

(A3) No interior vertex is connected to two or more non-consecutive vertices on the boundary of T.

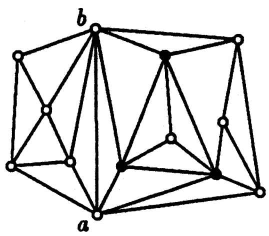

Figure 2.1: Chords and complex triangles: ab is a chord, and the three dark vertices form the boundary of a complex triangle.

Figure 3.1: A planar point set that does not admit a noncomplex triangulation.

3 Characterizing point sets admitting noncomplex triangulations

We assume throughout this section that our point sets satisfy the general-position assumption introduced in Section 1: no three points are collinear. We also assume that all point sets have at least four points.

The planar point set S shown in Figure 3.1 does not admit a noncomplex triangulation. Indeed, in any triangulation of S, vertex x must be connected to every other vertex, as are consecutive vertices around the convex set $S-\{x\}$. Any triangulation of S must also contain a chord of the convex hull of $S-\{x\}$. This chord and the two edges joining its endpoints to x form a complex triangle.

Theorem 3.1 states that the example of Figure 3.1 is essentially the only 3-connectible planar point set that does not admit a noncomplex triangulation. The proof of Theorem 3.1 is constructive, and leads to an $O(n \log n)$ algorithm for constructing a noncomplex triangulation if one exists.

The following definition captures the salient properties of the example of Figure 3.1. A planar point set is *anomalous* if it contains a point x such that the following properties hold:

(B1) S has exactly three extreme vertices, one of which is x.

(B2) The set $S-\{x\}$ consists of the vertices of a convex polygon, P.

Theorem 3.1 If S is a planar point set in general position, then S admits a noncomplex triangulation if and only if (1) it is not anomalous, and (2) it is not the set of vertices of a convex polygon.

Theorem 3.2 If S is a planar point set in general position, with exactly three points on the convex hull boundary, then S is 4-connectible if and only if it is not anomalous.

Theorem 3.2 is an immediate consequence of Theorem 3.1 and Lemma 2.2. The necessity of conditions (1) and (2) in Theorem 3.1 follows from Lemma 2.1 and the preceding discussion of Figure 3.1. To establish sufficiency of these conditions, we show how to construct a noncomplex triangulation of a planar point set satisfying conditions (1) and (2).

Our construction relies heavily on the *convex layer* structure [1]. The outermost convex layer of S is the boundary of the convex hull, and each subsequent convex layer is defined recursively, to be the boundary of the convex hull of the set obtained by removing the vertices of all previously defined convex layers from S. We let k be the number of convex layers, with layer 1 the outermost layer and layer k the innermost layer. All layers except the innermost layer consist of convex polygons. The innermost layer may consist of either a single point, a line segment, or a convex polygon. If $k = 1$, S forms the vertices of a convex polygon, so we may assume that $k \geq 2$.

Our construction also involves adding edges between two consecutive layers derived from the convex layer structure to create a triangulation of the region between the two layers. The edges that have an endpoint on each of the two layers are called *cross edges*.

Consider the following strategy: compute the convex layers of S, triangulate the innermost layer (if it is a convex polygon), and compute a triangulation of each region between each two consecutive convex layers using only cross edges. If we attempt to use this strategy to produce a noncomplex triangulation, it can fail in one of three ways. First, a chord in the innermost layer may participate in a complex triangle. Second, one of the triangulations between regions may produce a complex triangle consisting of two inter-layer edges and one edge from the inner or outer layer. Third, if any intermediate convex layer is a triangle, the strategy clearly fails. With appropriate modification, however, these difficulties can be overcome for non-anomalous point sets.

The following lemma allows us to extend noncomplex triangulations from one layer to the next layer.

Lemma 3.3 Let A and B be two polygonal layers satisfying the following properties: (i) A is convex containing B inside where $|B| > 3$, (ii) The region between A and B is triangulated with cross edges, (iii) The polygon with the boundary B is triangulated without any complex triangle, and there is no complex triangle incident on a chord of B. Then a non-complex triangulation of the region inside A can always be constructed.

Assume $k \geq 3$. For $j = 1, \ldots, k-2$, define a *good level-j triangulation* to be a triangulation for which the following properties hold: (i) The boundary of the outer face consists of layer j, (ii) All points of S inside the boundary are vertices of the triangulation, (iii) The triangulation contains no chords or complex triangles. The proof of Theorem 3.1 for $k \geq 3$ follows immediately from the following two lemmas, since a noncomplex triangulation is simply a good level-1 triangulation:

Lemma 3.4 S has a good level-$(k-2)$ triangulation.

Lemma 3.5 Given a good level-j triangulation, for $2 \leq j \leq k-2$, we can construct a good level-$(j-1)$ triangulation.

Proof of Lemma 3.4: Let A be layer $k-2$, B layer $k-1$, and C layer k. We distinguish six cases, which we group as follows. B may be either a polygon having more than three vertices (Part I) or a triangle (Part II). Within each part, C may consist of a single vertex ($|C| = 1$), a line segment ($|C| = 2$), or a convex polygon ($|C| > 2$).

Part I: B is not a triangle: We first obtain a good triangulation of the region inside B using the appropriate case (1, 2, or 3) below. We then extend this triangulation to a good triangulation of the region inside A using Lemma 3.3.

Case 1, where $|C| = 1$ and Case 2, where $|C| = 2$ can be handled easily. We omit the details.

Case 3: $|C| \geq 3$: C is the boundary of a convex polygon, inside B. First, compute some arbitrary triangulation of the region between C and B using cross edges. Let R be the set of vertices of C that are connected to two or more vertices of B. R contains at least two vertices of C. There are two subcases, depending on whether R contains two nonconsecutive vertices of C.

Subcase 3a: R contains two nonconsecutive vertices of C. Let c_1 and c_2 be a pair of nonconsecutive vertices of C in R (see Figure 3.2(a)). Removing these two vertices from the boundary of C creates two nonempty arcs, C' and C'', so no vertex of B can be joined to a vertex of C' and a vertex of C''. Hence if C is triangulated using only edges with one endpoint on C' and the other on C'' (in other words, triangulated so that c_1 and c_2 are ears), then no such edge can participate in a complex triangle. This produces a triangulation T containing layers B and C that satisfy all conditions of Lemma 3.3. Hence, T can be modified to be a non-complex triangulation.

Subcase 3b: R contains only a single pair of consecutive vertices of C, c_1 and c_2. This subcase is illustrated in Figure 3.2(b). Then all vertices of C are joined to a single vertex x of B. Assume that c_2 is the counterclockwise neighbor of c_1 on C; let c_1' be the clockwise neighbor of c_1, c_2' the counterclockwise neighbor of c_2. If C is a triangle, then $c_1' = c_2'$, but that does not affect the following argument. Now let x_1 and x_2 be, respectively, the counterclockwise and clockwise neighbors of x about the boundary of B. If the segment x_1c_1' does not intersect the interior of polygon C, then we can obtain a good triangulation by flipping edge xc_1, and triangulating the interior of C by joining c_1 to every vertex of C. If the segment x_2c_2' does not intersect the interior of polygon C, a similar construction works. Suppose neither of these last two conditions holds; then segments x_1c_1' and x_2c_2' both intersect segment c_1c_2. Let x' be the counterclockwise neighbor of x_1 about B. Since B is not a triangle, $x' \neq x_2$. Since B is convex, a good triangulation can be obtained by deleting edge c_1c_2 and connecting x' to every vertex of C.

Once the appropriate case (1, 2, or 3) is handled, use cross edges to triangulate the region between A and B. Apply Lemma 3.3, if necessary, to produce a good level-$(k-2)$ triangulation.

Part II: B is a triangle: Let $B = b_1b_2b_3$. We need to delete at least one edge of B. Consider the arrangement of the three lines that support the three edges of B. This arrangement has seven planar regions: the triangular region B, and six other regions exterior to the triangle. Call the three exterior regions bounded by three lines *type-1* regions, and the three exterior regions bounded by two lines *type-2* regions.

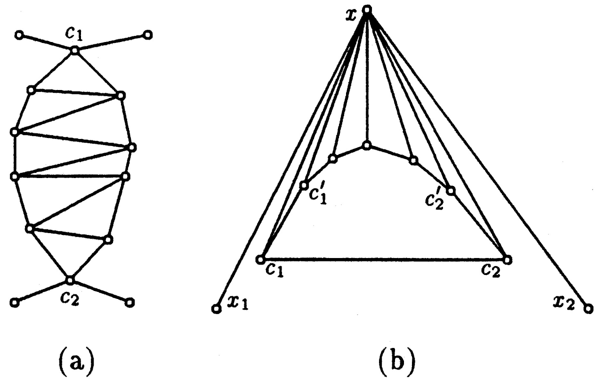

Figure 3.2: Obtaining a good triangulation when the first two layers are convex polygons. (a) Subcase 3a: c_1 and c_2 are two nonconsecutive vertices of the inner polygon that have two neighbors on the outer polygon. (b) Subcase 3b: c_1 and c_2 are consecutive, and are the only two vertices of the inner polygon that have two neighbors on the outer polygon.

We borrow one vertex p from A to augment B to a quadrilateral, B'. If A contains any vertex in a type-1 region, we show that we can choose p to be a vertex in a type-1 region. This means that the resulting quadrilateral is convex. If there is no vertex in any of the type-1 regions, we choose p from one of the type-2 regions. In this case the resulting quadrilateral will be non-convex only at a single vertex. It can be shown that a good triangulation of B' can be constructed which can be extended to a triangulation of A. We omit the details here. □

Proof of Lemma 3.5: Using constructions of Case 3 and Lemma 3.3 we can extend a good level-j triangulation to a good level-$(j-1)$ triangulation. □

Proof of Theorem 3.1: To complete the proof of Theorem 3.1, it suffices to address the case $k = 2$. If the outer layer has 4 or more points, the existence of a noncomplex triangulation follows immediately from the constructions in Part I of Lemma 3.4. So suppose the outer layer has 3 points. If there is exactly one point inside, the unique possible triangulation is noncomplex. If there are exactly two points inside, the configuration is anomalous. The remaining case where the inner layer consists of a convex polygon and the configuration is not anomalous can be handled using constructions of Lemma 3.4. Details are omitted. □

The following two theorems are easy consequences of our constructions.

Theorem 3.6 A planar point set S in general position can be augmented using at most two extra points so that it admits a 4-connected triangulation.

Theorem 3.7 Given a planar point set S in general position, in $O(n \log n)$ time we can either construct a non-complex triangulation of S if it admits one, or report that no such triangulation exists.

4 Conclusions and open problems

In this paper we have characterized the point sets that admit a noncomplex triangulation. This solves the question of 4-connectibility for a point set with three extreme vertices. However, it does not solve the 4-connectibility problem in general, and this problem remains open.

Figures 4.1 and 4.2 illustrate two ways that a planar point set can fail to be 4-connectible. The set shown in Figure 4.1 fails to be 4-connectible because there are fewer interior points than convex hull edges. Any triangulation of this point set must either have a chord, violating condition(A1) of Lemma 2.2, or two triangles having distinct convex hull edges as their bases but sharing an interior point as their common apex. In the latter case, condition (A3) of Lemma 2.2 is violated. Figure 4.2 also fails to be 4-connectible, even though it has more interior points than convex hull edges. To see this, note that if a 4-connected triangulation of this set exists, then one of the circled points (call it p) would have to be connected to y; otherwise x and z would have a common interior neighbor, violating (A3). But then p is connected to both w and y, so (A3) fails anyway.

We have not addressed the condition of 5-connectibility. It follows from the results in [4] that a triangulation is 5-connected if it satisfies conditions (A1)-(A3), has no complex (i.e., nonfacial) quadrilateral, and has no interior edge connected to two or more nonconsecutive boundary vertices. A simpler problem than general 5-connectibility might be characterizing those planar point sets that admit triangulations without complex quadrilaterals.

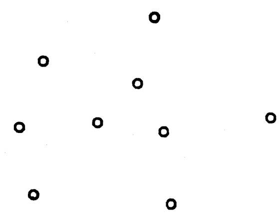

Figure 4.1: A point set that admits a noncomplex triangulation but is not 4-connectible.

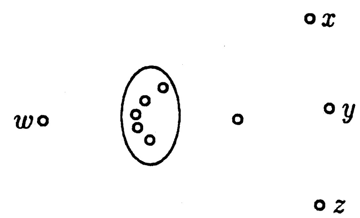

Figure 4.2: A point set that admits a noncomplex triangulation but is 4-connectible, even though it has more interior points than convex hull points.

Finally, we briefly discuss the general position assumption made in this paper, namely that no three points are collinear. There are non-anomalous point sets, not in general position, that do not admit a non-complex triangulation. One such point set is obtained by placing points on three lines meeting at the origin.

References

[1] B. Chazelle. On the convex layers of a planar set. *IEEE Transactions on Information Theory*, 31(4):509–517, July 1985.

[2] G. Di Battista, P. Eades, R. Tamassia, and I. G. Tollis. Algorithms for drawing graphs: An annotated bibliography. Manuscript, June 1993.

[3] M. B. Dillencourt. Hamiltonian cycles in planar triangulations with no separating triangles. *Journal of Graph Theory*, 14(1):31–49, 1990.

[4] J-P. Laumond. Connectivity of planar triangulations. *Information Processing Letters*, 34(2):87–96, March 1990.

[5] K. Yeap and M. Sarrafzadeh. Floor-planning by graph dualization: 2-concave rectilinear modules. *SIAM Journal on Computing*, 22(3):500–526, June 1993.