Cardinality Bounds for Triangulations with Bounded Minimum Angle

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Abstract

We consider bounding the cardinality of an arbitrary triangulation with smallest angle α. We show that if the local feature size (i.e. distance between disjoint vertices or edges) of the triangulation is within a constant factor of the local feature size of the input, then $N < O(1/\alpha)M$, where N is the cardinality of the triangulation and M is the cardinality of any other triangulation with smallest angle at least α. Previous results [7, 8] had an $O(1/\alpha^{1/\alpha})$ dependence. Our $O(1/\alpha)$ dependence is tight for input with a large length to height ratio, in which triangles may be oriented along the long dimension.

1 Introduction

We consider a triangulation used as a mesh for a finite element method. The important properties of such a triangulation are the shape and the number of its elements: Shape affects the accuracy of the numerical results, and the number of elements affects the running time. We measure shape by the smallest angle. Every triangulation must have an angle no larger than the smallest input angle, and this is tight up to small constant factors due to integrality [3, 8].

Typically many vertices are added to an input to produce a triangulation. Proving a lower bound on the number of vertices (cardinality) needed to achieve a given smallest angle is the main topic of this paper. By considering a long thin rectangle, it is obvious that the number of triangles necessary to ensure that all angles are at least some fixed α depends on the geometry of the input. Bern, Eppstein and Gilbert [1] compared the cardinality of their triangulation with the smallest angle in a Delaunay triangulation of the input. In three dimensions, Mitchell and Vavasis [7] were able to define the cardinality of their tetrahedralization in terms of a theoretical lower bound for the given input: Their tetrahedralization has cardinality N and smallest angle α, and any tetrahedralization with cardinality m and smallest angle at least α has $M > Nc(\alpha)$, where $c(\alpha)$ is a function of (only) α. This bound depends upon the notion of *local feature size*, distance to disjoint vertices or edges. Ruppert [8] used the same technique to bound triangulation cardinality in two dimensions, and showed that local feature size could be extended to a continuous function in the plane, related to the second-order Voronoi diagram [2].

Up until this present work, this approach had a serious flaw. The analysis of Ruppert [8] derives a $c(\alpha)$ that is very large, about 10^{25}, and reveals that the analysis of Mitchell and Vavasis [7] has c depending doubly exponentially on $1/\alpha$.

In this paper we show that the relationship between local feature size and cardinality is quite tight. In particular, we derive a $c(\alpha)$ depending *linearly* on $1/\alpha$ which is tight up to (reasonable) constant factors. This improves the bounds on the algorithm of Ruppert [8]. This is also in good agreement with the intuition of practitioners: Advancing front algorithms often use a notion of local feature size when deciding the size of elements to introduce [4].

1.1 Overview

Local feature size at a point z, $\text{lfs}(z)$ roughly measures the largest possible size of a triangle containing z in a triangulation with all triangles nearly equilateral. In Section 3 we show that if triangles with angle α are allowed, then an edge of a triangle containing z in a valid triangulation has length $O(\text{lfs}(z)2^{1/\alpha})$. We relate this to the extent of a Voronoi cell, and show that the integral of $1/lfs^2$ over points near a vertex of a triangulation scales only linearly with $1/\alpha$. For PSLG and polygon with holes input, we prove a similar result for points near an edge of the triangulation. By integrating over the entire input, we obtain a lower bound on the cardinality of a (theoretically best) triangulation. In Section 4 we show that an (algorithmically generated) triangulation has cardinality at most

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a constant factor times its local feature size integral (independent of α). Thus to prove that an algorithm produces a triangulation with reasonable cardinality, one need only show that the triangles produced are large compared to the local feature size of the input.

2 Definitions

We consider several types of input; point sets, planar straight-line graphs, and polygons with holes. Each of these we denote by P.

Local feature size. We define *local feature size* in P at a point z, or $\text{lfs}_P(z)$, as the radius of the smallest circle centered at z that contains points of disjoint faces of P. Most often we will be concerned with local feature size defined by the faces of a triangulation, $\text{lfs}_{\mathcal{T}}$, which is necessarily smaller than lfs_P. By faces we mean vertices and edges for PSLG or polygon with holes input, and just vertices for point set input. We call the integral of $1/\text{lfs}^2$ the *local feature size integral.*

Voronoi cell. We make use of the Voronoi diagram of the vertices of a triangulation [2]. The Voronoi diagram partitions the input into a set of convex cells. Each cell $\mathcal{V}\text{or}(V)$ consists of the points closer to the given vertex V than any other vertex.

3 Lower bounds on cardinality

Here we show that any triangulation with minimum angle bounded by α must have at least a certain number of vertices, depending on the local feature size of the input and linearly on $1/\alpha$.

In a triangulation with bounded smallest angle, the longest length of an edge at a vertex is bounded in terms of $k^{1/\alpha}$ and the shortest edge at the vertex, where $k = 2\cos\alpha$, $1 \leq k < 2$. We extend this to bound the maximum extent of a Voronoi cell in terms of $k^{1/\alpha}$ and the minimum extent of the cell. Local feature size at a point in a Voronoi cell can be bounded by the the point's distance to the Voronoi cite, or by the minimum extent of the cell. For PSLG or polygon with holes input, we also consider zones for edges akin to the Voronoi cells for vertices. We then show that the local feature size integral over a cell or zone is $O(1/\alpha)$. Integrating over all of the triangulation shows that the cardinality of any triangulation times $O(1/\alpha)$ is larger than the local feature size integral over P.

We first bound the ratio of the length of an edge at a vertex in terms of the length of the smallest edge at that vertex; see Figure 1.

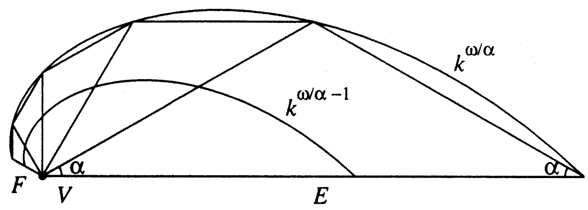

Figure 1: We bound how quickly triangles and Voronoi cells can grow.

Theorem 1 *At any vertex V of a triangulation with all angles at least α, we have*

$$\frac{|E|}{|F|} \leq k^{\frac{\angle EF}{\alpha}},$$

where E and F are edges at V, and $k = 2\cos\alpha$. Note $1 \leq k < 2$ and $\angle EF/\alpha \geq 1$.

Proof. We use induction on the number of edges between E and F at V. The base case is if there are no edges between E and F, that is if E and F are in a common triangle T. Let G be the third edge and e, f and g the angles opposite E, F and G.

From the law of sines, $|E|/|F| = \sin e/\sin f$. This may be expressed as $\sin(g + f)/\sin f = \cos g + \sin g \cos f/\sin f$. For any triangle angle θ we have $\pi > \theta \geq \alpha$, which implies $\cos\theta \leq \cos\alpha$. Hence the above is less than $\frac{k}{2}(1 + \frac{\sin g}{\sin f})$.

If $f > g$, then this is less than k. Otherwise, the worst case is when $f = \alpha$. If $g = \alpha$ as well, then the above equals k. Furthermore, since $\sin(g + \alpha)/\sin\alpha$ is a more slowly growing function of g than is $k^{g/\alpha}$, we have that $\sin(g + \alpha)/\sin\alpha < k^{g/\alpha}$ for all $g > \alpha$.

For the induction step, let H be any edge between E and F. By induction the theorem is true for the number of edges between E and H and between H and F. Thus $\frac{|E|}{|F|} = \frac{|E|}{|H|}\frac{|H|}{|F|} \leq k^{\angle EH} k^{\angle HF} = k^{\angle EF}$. ∎

Note that this is a contrapositive version of the key theorem of Mitchell [5, 6]. Our current theorem has two advantages. First, it is tight for integer $\angle EF/\alpha$. Second, its proof is much simpler than the seven page proof of Mitchell [6].

We may extend to the following theorem bounding the extent of a Voronoi cell; again see Figure 1.

Theorem 2 *For every point $z \in \mathcal{V}\text{or}(V)$, $\frac{\text{dist}(z,V)}{|F|} < k^{\frac{\omega}{\alpha}-1}$, where F is any edge containing V and ω is the angle between $\overline{zV}$ and F.*

Proof. In the given triangulation, z lies in the sector defined by two consecutive edges $\overline{VW}$ and $\overline{VX}$ at V, where W is the vertex that lies in the sector zVF.

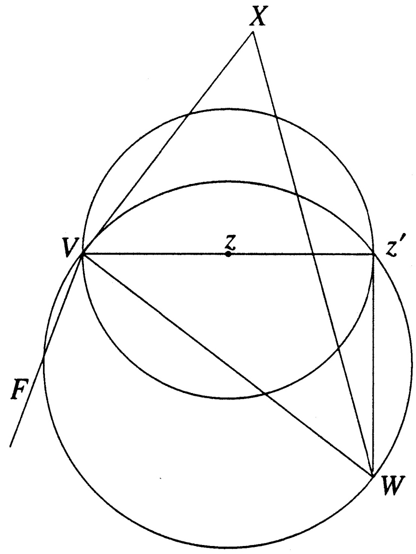

Figure 2: Bounding $\text{dist}(z, V)$.

We have two cases. If $\angle zVW < \alpha$ then the fact that z is closer to V than to W implies $\overline{zV}$ is short compared to $\overline{VW}$. Otherwise, we we replace ΔXVW with $\Delta z'VW$ (z' defined below, see also Figure 2) and show that this triangle has all angles at least α, so that Theorem 1 applies.

Suppose $\angle zVW < \alpha$. Since $z \in \mathcal{V}\text{or}(V)$ is closer to V than W, we have $\text{dist}(z, V) < \text{dist}(V, W)/2\cos(\angle zVW) < \text{dist}(V, W)/k$. Thus

$$\frac{\text{dist}(z, V)}{|F|} = \frac{\text{dist}(z, V)}{\text{dist}(V, W)} \frac{\text{dist}(V, W)}{|F|}$$

$$< k^{\frac{\omega - \angle zVW}{\alpha} - 1} < k^{\frac{\omega}{\alpha} - 1}.$$

Otherwise, replace ΔVWX by $\Delta z'VW$, where z' lies on the ray from V to z and $\text{dist}(z', V) = 2\text{dist}(z, V)$; see Figure 2. By assumption $\angle zVW = \angle z'VW \geq \alpha$. Now X lies outside the circle with diameter $\overline{z'V}$, hence outside the circle with diameter $\overline{VW}$, so $\alpha \leq \angle VXW < \angle Vz'W$. If $\text{dist}(z', V) \leq |\overline{V, W}|$ then the theorem easily follows by an argument similar to the first case. Otherwise $\angle z'VW \geq (\pi - \angle XVW)/2 \geq \alpha$. Thus $\Delta z'VW$ has all angles at least α, and hence Theorem 1 applies: $\text{dist}(z, V) = 0.5\text{dist}(z', V) \leq 0.5|F|k^{\frac{\omega}{\alpha}} < |F|k^{\frac{\omega}{\alpha}-1}$. ∎

3.1 Point set input

For point set input, we now relate local feature size to the minimum and maximum extent of a Voronoi cell. Let lfs_T denote the local feature size defined by the vertices T of the triangulation under consideration. Since $T \supseteq P$, $\text{lfs}_T \leq \text{lfs}_P$.

Definition l**.** Let l denote the shortest possible length of an edge F at V from Theorem 1, given the longest edge E.

Theorem 3 *For point set input, any point* $z \in \mathcal{V}\text{or}(V)$ *has* $\text{lfs}_T(z) \geq \max(\text{dist}(z, V), l/2)$.

Proof. Consider the circle $\oslash$ defining local feature size at z. Then V lies inside $\oslash$. Hence $\text{lfs}_T(z) \geq \text{dist}(z, V)$. Also another vertex V' lies on the boundary of $\oslash$. Hence $2\text{lfs} \geq \text{dist}(V, V')$. We may replace the given triangulation with the Delaunay triangulation (DT) [2] of its vertices. This does not decrease the smallest angle (the DT maximizes the minimum angle), and places an edge between V and V' (the DT places an edge between vertices that share an empty circle). Since l is defined to be the theoretical minimum edge length, and there is now an actual edge between V and V', we have $\text{dist}(V, V') \geq l$. ∎

Theorem 4

$$\int_{z \in \mathcal{V}\text{or}(V)} \frac{dz}{\text{lfs}^2(z)} \leq \frac{\pi^2 \ln k}{\alpha} + \pi(1 + 2\ln(2/k))$$

$$\leq \frac{6.9}{\alpha} + 7.5.$$

Proof. We integrate radially about V. From Theorem 2 we have an upper bound on the distance of any point in the cell to V. And Theorem 3 bounds lfs below. Hence

$$\int_{z \in \mathcal{V}\text{or}(V)} \frac{dz}{\text{lfs}_T^2(z)} \leq 2\int_{\theta=0}^{\pi} \int_{r=0}^{lk^{\theta/\alpha - 1}} \frac{r\,dr\,d\theta}{\max^2(r, l/2)}$$

$$= 2\pi \int_0^{l/2} \frac{r\,dr}{l^2/4} + 2\int_0^{\pi} \int_{l/2}^{lk^{\theta/\alpha - 1}} \frac{dr\,d\theta}{r}$$

$$= \pi + 2\int_0^{\pi} \left(\frac{\theta \ln k}{\alpha} + \ln(2/k)\right) d\theta$$

$$= \frac{\pi^2 \ln k}{\alpha} + \pi(1 + 2\ln(2/k)).$$

∎

By summing the integral over all Voronoi cells we get the following.

Theorem 5 *Any point set triangulation with smallest angle at least* α *has at least* M *vertices with*

$$M\left(\frac{6.9}{\alpha} + 7.5\right) > \int_P \frac{1}{\text{lfs}^2}.$$

Note that the linear tradeoff between M and $1/\alpha$ is tight for the vertices of a long, thin rectangle and α greater than the small angle between the diagonals of the rectangle.

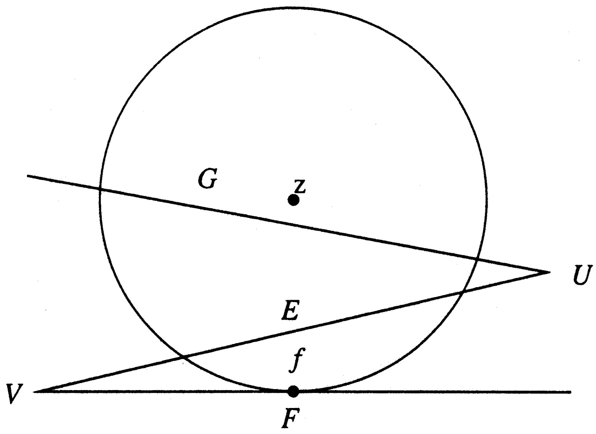

Figure 3: Point z lies in the zone for edge E.

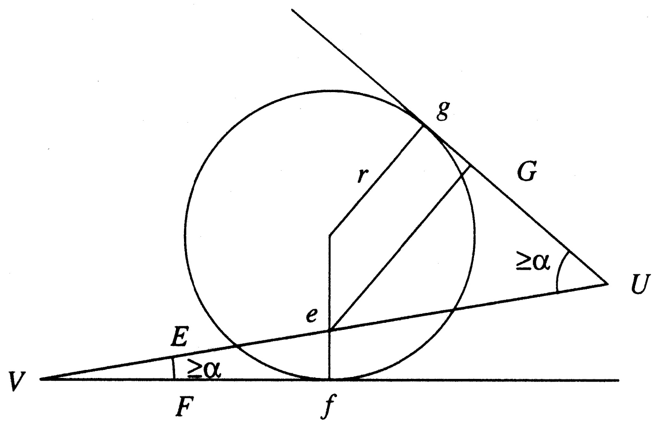

Figure 4: $\text{lfs}_{\mathcal{T}}(z) \geq r \geq |E| \sin \alpha$.

3.2 PSLG and polygon with holes input

We now consider planar straight-line graph (PSLG) and polygon with holes input. As before we consider the local feature size determined by the triangulation, $\text{lfs}_{\mathcal{T}}$, which is not larger than that determined by the input, $\text{lfs}_{\mathcal{P}}$. However, local feature size is now determined by disjoint edges as well as vertices. Hence we must consider the case that local feature size at a point is determined by two disjoint edges. Because of this we define zones for edges as well as use the Voronoi cells of the previous section. (Voronoi cells define vertex zones, but the medial axis [2] bears little resemblance to edge zones because edges must be disjoint to determine feature size.)

Edge zone. We place a point in an edge zone if its local feature size circle $\oslash$ does not contain a vertex. In this case, there will always be an edge $E = \overline{UV}$ piercing $\oslash$ such that an edge F containing V and an edge G containing U also pierce $\oslash$; see Figure 3. (Proof: Since we have a triangulation, neighboring edges in $\oslash$ must share a vertex. If every edge shares a common vertex, then there are no disjoint faces in $\oslash$.) We place z in the edge zone for E.

Theorem 6 *For a point $z \in \text{zone}(E)$, $\text{lfs}_{\mathcal{T}}(z) \geq \max(\text{dist}(z, E), |E| \sin\alpha/2)$.*

Proof. That $\text{lfs}_{\mathcal{T}}(z) \geq \text{dist}(z, E)$ is obvious from the fact that E passes through $\oslash$, whose radius is $\text{lfs}_{\mathcal{T}}(z)$.

For the second relation, the intuition is that the circle contains most of the altitude of one of the triangles containing E. One of the edges, say F, will be tangent to $\oslash$ at f; see Figure 3. Let C be the circle tangent to F at f and tangent to G. Let $r \leq \text{lfs}(z)$ denote its radius; see Figure 4. Let e be the point of E that crosses either the radius to G or to F of C. From the triangle inequality $2r \geq \text{dist}(e, F) + \text{dist}(e, G)$, where F and G now represent the lines through F and G if necessary. But also $\text{dist}(e, F) + \text{dist}(e, G) = \text{dist}(e, V) \sin \angle EF + \text{dist}(e, U) \sin \angle EG \geq |E| \sin \alpha$. ∎

Theorem 7 *For an edge E,*

$$\int_{z \in \text{zone}(E)} \frac{1}{\text{lfs}_{\mathcal{T}}^2} < \frac{4}{\sin \alpha}.$$

Proof. From Theorem 6 we have $\int_{z \in \text{zone}(E)} \frac{1}{\text{lfs}_{\mathcal{T}}^2} \leq \int_{z \in \text{zone}(E)} \frac{1}{\max^2(\text{dist}(z,E), |E| \sin\alpha/2)}$, and if we integrate along the length of E and out to infinity,
$< \int_0^{|E|} \left(\int_0^{|E| \sin\alpha/2} \frac{4}{|E|^2 \sin^2 \alpha} + \int_{|E| \sin \alpha/2}^{\infty} \frac{1}{\text{dist}^2(z,E)} \right)$
$= |E| \left(\frac{2}{|E| \sin\alpha} + \frac{2}{|E| \sin \alpha} \right)$. ∎

Relating local feature size to the minimum extent of a Voronoi cell is more complicated than in the point set case because the edges of triangles opposite a vertex contribute to local feature size, but not to the Voronoi cell (considering the medial axis directly does not appear helpful).

Theorem 8 *For any triangle with vertex V and opposite edge E, $\text{dist}(E, V) \geq l \cos \alpha$. Recall l is the lower bound on the minimum possible length of an edge F at V from Theorem 1, given the actual longest edge E at V.*

Proof. If the closest point of E to V is a vertex, then their distance is an edge of the triangle, which by definition has length at least l. Otherwise, their distance is defined by the altitude A at V. Let F be the shorter triangle edge containing V. We have two subcases. If the angle between A and F is less than α, then $|A| = |F| \cos \angle AF \geq l \cos \alpha$. Otherwise, we may add A as an edge of the triangulation and still ensure all angles at least α (triangles not containing V may be ignored). This implies $|A| \geq l$ from Theorem 1. ∎

Theorem 9 *For PSLG or polygon with holes input, for a point* $z \in \mathcal{V}or(V)$, $\mathrm{lfs}_{\mathcal{T}}(z) \geq \max(\mathrm{dist}(z, V), l\cos\alpha/2)$.

Proof. By definition, the local feature size circle $\oslash$ contains V, so the first relation is obvious. Since $\oslash$ is the smallest circle at z containing a face disjoint from V, we have that $\oslash$ contains a point of a triangle edge opposite V. Hence $2\mathrm{lfs}_{\mathcal{T}} \geq \mathrm{dist}(V, E)$. By Theorem 8 we have $\mathrm{dist}(V, E) \geq l\cos\alpha$. ∎

Theorem 10 *For PSLG input,*

$$\int_{z\in\mathcal{V}or(V)} \frac{dz}{\mathrm{lfs}_{\mathcal{T}}^2(z)} < \frac{\pi^2 \ln k}{\alpha} + \pi(1 + 2\ln(2/k\cos\alpha))$$

$$< \frac{6.9}{\alpha} + 11.9.$$

For polygon with holes input,

$$\int_{z\in\mathcal{V}or(V)} \frac{dz}{\mathrm{lfs}_{\mathcal{T}}^2(z)} < \frac{2\pi^2 \ln k}{\alpha} + \pi(1 + 2\ln(2/k\cos\alpha))$$

$$< \frac{13.7}{\alpha} + 11.9.$$

Proof. Replace the use of Theorem 3 with Theorem 9 in the proof of Theorem 4. For polygon with holes input, the angle between two edges at an input vertex may be obtuse, up to 2π. ∎

We may bound the local feature size integral over P by summing the local feature size integrals over the vertex and edge zones of the triangulation. Since by Euler's theorem there are at most three times as many vertices as edges, we may combine Theorem 7 and Theorem 10:

Theorem 11 *Given a PSLG P, any triangulation with all angles at least* α *has at least M vertices, with*

$$M(\frac{21.5}{\alpha} + 11.9) > \int_P \frac{1}{\mathrm{lfs}_P^2}.$$

For polygon with holes P, we have

$$M(\frac{43.0}{\alpha} + 11.9) > \int_P \frac{1}{\mathrm{lfs}_P^2}.$$

Note that as in the point set case the linear tradeoff between M and $1/\alpha$ is tight for a long, thin rectangle and α greater than the small angle between the diagonals of the rectangle.

4 Upper bounds on cardinality

We now show that the number of vertices in an (algorithmically generated) triangulation $\mathcal{T}$ is at most a constant factor times the integral of $1/lfs_{\mathcal{T}}^2$. We use this to show that an algorithmically generated triangulation is small if its local feature size is large compared to the local feature size of the input. The following theorems hold for point set, PSLG and polygon with holes input. Recall that $\mathrm{lfs}_{\mathcal{T}}$ is defined by the faces of the triangulation, either its vertices (for point set input) or by its vertices and edges (for PSLG or polygon with holes input).

Theorem 12 *For any non-input vertex V,*

$$\int_{\mathcal{V}or(V)\cap P} \frac{1}{\mathrm{lfs}_{\mathcal{T}}^2} > 0.226$$

Proof. Let $R = \mathrm{lfs}_{\mathcal{T}}(V)$ be the distance from V to the closest point on a face disjoint from V. From the triangle inequality $\mathrm{lfs}_{\mathcal{T}}(z) \leq \mathrm{dist}(z, V) + R$. Since V is not an input vertex, it must lie interior to P or in the relative interior of an edge on the boundary of P. Hence $\mathcal{V}or(V) \cap P$ must contain a semi-circle of radius $R/2$. Thus

$$\int_{\mathcal{V}or(V)\cap P} \frac{1}{\mathrm{lfs}_{\mathcal{T}}^2} \geq \int_0^{\pi} \int_0^{R/2} \frac{r dr d\theta}{(r + R)^2}$$

$$\geq \pi(\ln\frac{3}{2} - \frac{1}{3}) > 0.226.$$

∎

We may easily extend to the following:

Theorem 13 *Any triangulation* $\mathcal{T}$ *has at most* N' *non-input vertices, with*

$$0.226N' < \int_P \frac{1}{\mathrm{lfs}_{\mathcal{T}}^2}.$$

Combining the results of the two sections we have:

Theorem 14 *Suppose a triangulation* $\mathcal{T}$ *with smallest angle* α *has* $\mathrm{lfs}_{\mathcal{T}} \geq k_1 \mathrm{lfs}_P$, *then the cardinality of* $\mathcal{T}$ *is less than* k_2 *times the cardinality of any other triangulation with smallest angle at least* α, *where* $k_2 = k_1^2 O(1/\alpha)$.

Proof. From Theorem 4, Theorem 11, and Theorem 13, $N' < k_3 M$ with $k_3 = O(1/\alpha)$, where M is the cardinality of any triangulation with smallest angle at least α and N' is the number of non-input vertices of $\mathcal{T}$. But $|\mathcal{T}| = N = N' + n < k_3 M + n \leq (k_3 + 1)M = k_2 M$. ∎

Note $k_2 < k_1^2(\frac{30.5}{\alpha} + 34.2)$ for point set input, $k_2 < k_1^2(\frac{95.1}{\alpha} + 53.7)$ for PSLG input and $k_2 < k_1^2(\frac{190.2}{\alpha} + 53.7)$ for polygon with holes input.

In Ruppert's PSLG triangulation algorithm [8] we can show that for $\alpha = \pi/9$, $k_2 = 6.3\text{x}10^5$ (21.7 from Theorem 11, and 29,000 from Ruppert [8]). For $\alpha = \pi/18$, $k_2 = 9.0\text{x}10^3$ (41.66 from Theorem 11, and 215 from Ruppert [8]). This is quite large, and might possibly be improved with better analysis of Ruppert [8], but it still is much better than the factor of $2\text{x}10^{25}$ found in Ruppert [8].

5 Conclusions

We have proven tight bounds on the cardinality of a triangulation in terms of local feature size and the smallest angle, up to constant factors. We have also shown that two triangulations with similar local feature size must have similar cardinality, up to a $1/\alpha$ factor. This factor is tight: between two parallel input edges, a triangulation consisting of equilateral triangles and a triangulation consisting of skinny triangles aligned with the edges have the same local feature size.

For future work, the results may be extended to higher dimensions. We conjecture that in three dimensions $c(\alpha) = O(1/\alpha)$ is tight for point set and other convex input, but that $c(\alpha) = O(1/\alpha^2)$ is tight for non-convex polytopes due to the possibility of fan-like edges emanating from a vertex.

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