On the Number of Extrema of a Polyhedron †

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Abstract: In this paper we obtain a tight bound on the number of local minima and maxima (collectively called *extrema*) of a three-dimensional polyhedron P with respect to an arbitrary direction λ. For simplicity, we assume that no edge of P is normal to λ, which implies that only vertices of P can be extrema; if we relax this assumption, our bound translates into a bound on the number of maximal connected sets of extrema. We prove that if the polyhedron has r reflex angles, then the number of extrema (with respect to a given direction) does not exceed $2r+2$. Moreover, for every integer $r \geq 0$, we exhibit a polyhedron that has r reflex angles and $2r+2$ extrema with respect to a certain direction.

1. Introduction.

Consider a three-dimensional polyhedron P and an arbitrary oriented line λ; we are interested in computing bounds on the number of *extrema* of P (that is, its local minima and maxima) with respect to λ. For simplicity, we assume that no edge of P is normal to λ, which implies that only vertices of P can be extrema; otherwise (in this case, an infinite number of points may contribute extrema with respect to λ), our bound translates into a bound on the number of maximal connected sets of extrema.

Except for its importance as a combinatorial result, a bound on the number of extrema proves to be useful in estimating the complexity of algorithms. For example, in two dimensions, a $2r+2$ upper bound on the number of extrema of a polygon that has r reflex vertices proves instrumental in the analysis of Hertel and Mehlhorn's algorithm for polygon triangulation [5]. In turn, our interest in the three-dimensional version of the problem is motivated from an algorithm to decompose the boundary of a polyhedron into "well behaved" patches [3], where the number of produced patches relies on the number of extrema of the polyhedron. It is important to observe that the number of extrema of a polygon or polyhedron is directly related to how "non-convex" the polygon or polyhedron is; after all, a convex object exhibits exactly two extrema. So, it is desirable that the bound on the number of extrema be expressed in terms of a measure of non-convexity of the polygon or polyhedron. For a polygon, it is the number of its reflex vertices that is most commonly used for this purpose; for a polyhedron it is the number of its reflex edges, i.e., edges whose corresponding (internal) dihedral angles exceed π. Several recent algorithms and bounds have been expressed in terms of these measures (see [4], [1], [2]).

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The problem of computing bounds on the number of extrema of a polygon is very simple: an easy inductive proof (it involves "resolving" a reflex vertex at the inductive step) establishes an $r+2$ upper bound, where r is the number of reflex vertices of the polygon; the bound is tight, since for every integer $r \geq 0$ there exists a polygon that has r reflex vertices and $r+2$ extrema (see Figure 1). This approach does not extend to the three-dimensional case, however; "resolving" a reflex edge may lead to splitting other reflex edges, thus effectively doubling their number in the worst case. Our approach relies on observing how the number of polygons changes in the intersection of a polyhedron P with a sweep plane that moves along a direction λ; it helps us prove an upper bound of $2r+2$ on the number of extrema of P with respect to λ; r denotes the number of reflex edges of P. We also show that this bound is tight by exhibiting a polyhedron that has r reflex edges and $2r+2$ extrema (for every integer $r \geq 0$).

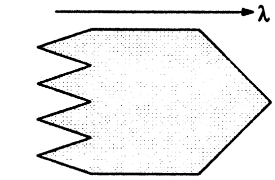


Figure 1

The paper is structured as follows. In Section 2 we introduce our terminology. The upper bound is proved in Section 3, while the lower bound is presented in Section 4. Finally, in Section 5 we summarize our results and pose some open questions.

2. Geometric Framework.

A polyhedron in $\mathcal{R}^3$ is a connected piecewise-linear 3-manifold with boundary. Its boundary is connected and it consists of a collection of relatively open sets, the *faces* of the polyhedron, which are called *vertices*, *edges*, or *facets*, if their affine closures have dimension 0, 1, or 2, respectively. The definition of a polyhedron rules out self-intersecting, dangling, or abutting faces, as well as degeneracies like the ones shown in Figure 2 (the shown objects have locally been compressed into a single point or a single edge). Then, each edge is incident upon exactly two facets; if the (interior) dihedral angle formed by the facets incident upon an edge e of a polyhedron exceeds π, we say that e is *reflex*.

Next, we formally define the notion of extrema in any dimension d. A point p of a d-dimensional set S of points is called an *extremum* of S with respect to an oriented line λ, or a *λ-extremum* for short, if the intersection of S with a small enough d-ball centered at p lies entirely in the one of the two closed halfspaces defined by the hyperplane normal to λ that passes through p. The extrema are characterized as *negative* or *positive* depending on whether the above intersection lies in the non-negative or non-positive halfspace respectively. For the polygon of Figure 3, for instance, the vertices A, B, C and D are negative λ-extrema, while X and Y are positive ones. The vertex Z is *not* an extremum. The definition allows for all the points of an edge or even a facet of a polyhedron P to be extrema, which obviously defeats the purpose of computing an upper bound on their number. If we assume, however, that no edge of P is normal to the line λ, then only vertices of P can be extrema, and hence their number is bounded; moreover, a bound on the number of extrema under the above assumption translates into a bound on the number of maximal connected sets of extrema if the assumption is relaxed.

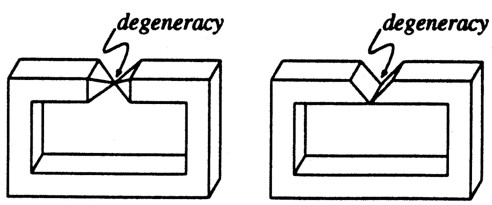


Figure 2

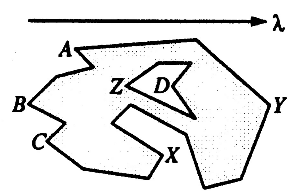


Figure 3

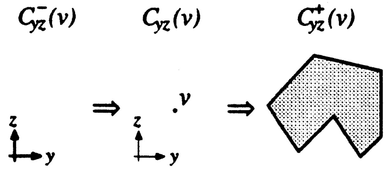


Figure 4

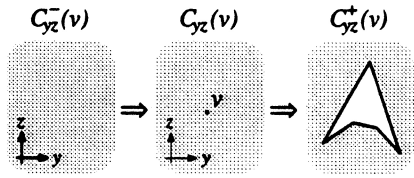


Figure 5

3. The Upper Bound.

Without loss of generality, we assume that we wish to compute the number of extrema with respect to the x-axis of a polyhedron P. We additionally assume that no edge of P is normal to the x-axis, so that we have a unique point-representative for every connected set of extrema; otherwise, we can use a lexicographic ordering, which essentially corresponds to an infinitesimal tilting of the polyhedron. The basic idea is to sweep P with a plane normal to the x-axis, and observe the changes in the intersection of P with the plane: we call such an intersection a *yz-cross-section* of P. It is important to observe that for as long as the plane sweeps the polyhedron without encountering a vertex, the corresponding yz-cross-section changes conformally; therefore, we need to concentrate on how the yz-cross-section changes, as the plane passes through a vertex; in particular, we need to focus only on the neighborhood of that vertex in the cross-section. For simplicity, we use $C_{yz}(v)$ to denote the yz-cross-section when the sweep plane is located at the vertex v of the polyhedron; similarly, $C^-_{yz}(v)$ $(C^+_{yz}(v)$ resp.) denotes the cross-sections infinitesimally before (after resp.) the vertex v is reached. Depending on the geometry of the neighborhood of v in $C_{yz}(v)$, we distinguish the following cases:

1. **The neighborhood of v in $C_{yz}(v)$ is a point-polygon** (Figure 4): Then, v is an x-extremum. Figure 4 depicts how the cross-section changes from $C^-_{yz}(v)$ to $C^+_{yz}(v)$, if v is a negative x-extremum; the case for a positive x-extremum is obtained by interchanging the figures for $C^-_{yz}(v)$ and $C^+_{yz}(v)$. The definition of the polyhedron implies that as the sweep plane passes through v, either *exactly one* new polygon appears, or *exactly one* polygon disappears in the cross-section.
2. **The neighborhood of v in $C_{yz}(v)$ is a point-hole in a polygon** (Figure 5): Then, v is not an x-extremum, but it is like the vertex Z in Figure 3. Figure 5 depicts one of the two basic cases that may arise; the other one results from Figure 5 after the figures for $C^+_{yz}(v)$ and $C^-_{yz}(v)$ have been interchanged. In this case, the number of polygons in the cross-section does not change; instead, either one new hole appears, or one hole disappears.
3. **The degree of v in $C_{yz}(v)$ is 2** (Figure 6): In this case, v belongs to a single polygon, which may exhibit corrugations in the neighborhood of v in $C^+_{yz}(v)$ (Figure 6 shows two examples). In any case, the total number of polygons in the cross-section does not change.
4. **The vertex v is of degree larger than 2 in $C_{yz}(v)$:** This is the really interesting case; this time, the neighborhood of v in $C_{yz}(v)$ consists of, say, w_v wedges ($w_v \geq 2$) that touch at v ($w_v = 4$

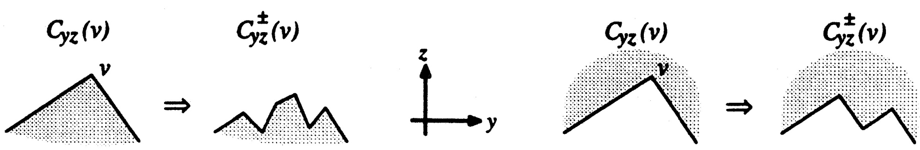


Figure 6

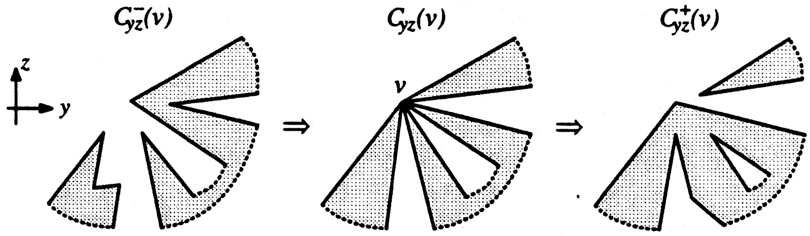


Figure 7

in Figure 7), and it is a combination of the two cases shown in Figure 8, as well as those obtained from Figure 8 in right-to-left order; the dashed curves indicate that some of the wedges may belong to the same polygon of $\mathcal{C}_{yz}(v)$. These wedges may either merge with or split from neighboring wedges in $\mathcal{C}^+_{yz}(v)$ and $\mathcal{C}^-_{yz}(v)$. In Figure 7 for instance, as $\mathcal{C}_{yz}(v)$ evolves into $\mathcal{C}^+_{yz}(v)$, the three bottom wedges merge into a single wedge-cluster, while the top wedge forms a wedge-cluster by itself. Let w^+_v and k^+_v (w^-_v and k^-_v) denote the number of such wedge-clusters and the number of polygons containing these clusters in $\mathcal{C}^+_{yz}(v)$ ($\mathcal{C}^-_{yz}(v)$) respectively (in Figure 7, we have $w^-_v = 3$, $w^+_v = 2$, and $k^-_v = k^+_v = 2$). Clearly,

$$w_v \geq w^+_v \geq k^+_v \quad \text{and} \quad w_v \geq w^-_v \geq k^-_v. \tag{1}$$

Moreover, the definition of a polyhedron (see Section 2) implies the following crucial observation:

Observation 1: if two wedges incident upon v in $\mathcal{C}_{yz}(v)$ merge in $\mathcal{C}^-_{yz}(v)$, they have to split in $\mathcal{C}^+_{yz}(v)$, and vice versa.

Otherwise, we end up with degeneracies like the one exhibited at the object on the left in Figure 2. In terms of w_v, w^-_v, and w^+_v, the observation can be expressed as follows (note that $(w_v - w^-_v)$ is equal to the number of wedges that get merged in $\mathcal{C}^-_{yz}(v)$):

$$(w_v - w^-_v) + (w_v - w^+_v) = w_v - 1 \iff w^-_v + w^+_v = w_v + 1. \tag{2}$$

A second crucial observation is:

Observation 2: if a vertex v is incident upon w_v wedges in $\mathcal{C}_{yz}(v)$, then it is incident upon *at least* $w_v - 1$ reflex edges of P.

This follows from the fact that when k wedges merge, they exhibit at least $k-1$ reflex edges at the merging "seams" (see Figure 7).

Depending on the case in which it falls, each vertex of P is put into one of four sets, say, V_1, V_2, V_3, and V_4 (corresponding to cases 1, 2, 3, and 4, respectively). Then,

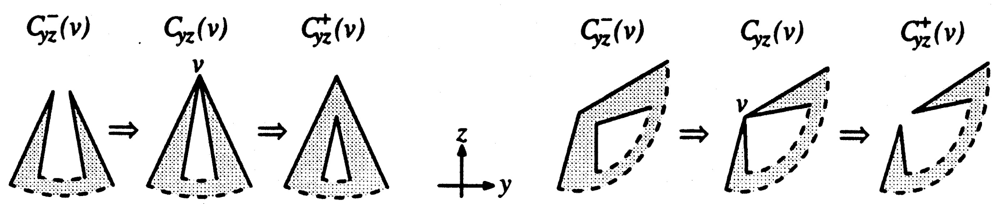


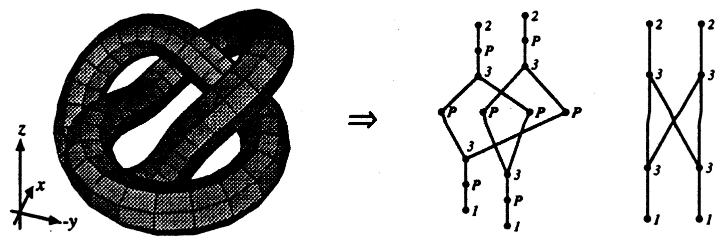
Figure 8

Lemma 3.1. *The total number $|V_1|$ of positive and negative x-extrema of a polyhedron does not exceed $\sum_{v \in V_4} (w_v - 1) + 2$, where w_v is as defined earlier.*

Proof: Again, we assume that we sweep the polyhedron P with a plane normal to the x-axis. We construct a graph that records the number of polygons in the history of the yz-cross-section; we concentrate only on the cases where the plane encounters positive or negative x-extrema, or vertices in V_4, since it is precisely then when the number of polygons changes. In particular,

1. at a negative x-extremum (vertex in V_1), we add to the graph two new nodes‡ which we connect by an edge; the first node corresponds to the negative x-extremum while the second one is a *polygon-node* and corresponds to the series of polygons to which the negative x-extremum evolves;
2. at a positive x-extremum (vertex in V_1), we add one new node that corresponds to the positive x-extremum and we connect it to the polygon-node that represents the polygon which shrunk to this positive x-extremum during the sweeping;
3. at a vertex v in V_4, we add one new node that corresponds to v and edges connecting it to the representatives of the k_v^- polygons in $C_{yz}^-(v)$. Moreover, k_v^+ polygon-nodes are added, one for each of the k_v^+ polygons in $C_{yz}^+(v)$, and edges are introduced between them and the node corresponding to v.

Since we are dealing with a single polyhedron, the resulting graph is connected. Figure 9(b) shows the graph that corresponds to the polyhedron of Figure 9(a). The numbers 1, 2, and 3 denote negative and positive x-extrema, and vertices in V_4 respectively (in agreement with the cases above), while the letter P denotes polygon-nodes. Note that all polygon-nodes are of degree 2, and no two of them are adjacent. Moreover, the nodes corresponding to negative and positive x-extrema of the polyhedron are of degree 1.



(a) (b) (c)

Figure 9

To simplify matters, we remove from the graph all polygon-nodes by coalescing their incident edges into a single edge; the resulting graph is a connected multigraph (see Figure 9(c)), whose node set is in one-to-one correspondence with the union of V_1 and V_4. If we denote by m the total number of edges of the multigraph, we have

$$2m = \sum_{\text{node } v} \text{degree}(v) = \sum_{v \in V_1} 1 + \sum_{v \in V_4} (k_v^- + k_v^+) \leq |V_1| + \sum_{v \in V_4} (w_v + 1), \tag{3}$$

since, for any vertex $v \in V_4$, inequalities (1) and (2) imply that $k_v^- + k_v^+ \leq w_v^- + w_v^+ = w_v + 1$. Connectivity, on the other hand, implies that the number of edges is at least equal to 1 less than the number of nodes of the graph, that is, $m \geq |V_1| + |V_4| - 1$, which combined with (3) yields

$$2\Big(|V_1| + |V_4| - 1\Big) \leq |V_1| + \sum_{v \in V_4} (w_v + 1).$$

The lemma follows. ∎

‡ We refer to the *nodes* of a graph (instead of vertices) to avoid confusion with the vertices of the polyhedron.

The desired upper bound on the number of extrema of a polyhedron follows as a consequence of Lemma 3.1 and Observation 2.

Theorem 3.1. *The total number of positive and negative x-extrema of a polyhedron with r reflex edges does not exceed* $2r+2$.

4. The Lower Bound.

The above upper bound is in fact tight, since there is a polyhedron with r reflex edges that has $2r+2$ extrema (for arbitrary r). Our construction involves the basic polyhedron shown in Figure 10: it looks like a house whose corners u and v have been pulled outwards along the direction λ; u and v are the extrema with respect to λ. By gluing $r+1$ of these polyhedra together at their shaded facets, we get a polyhedron with r reflex edges and $2(r+1)$ extrema with respect to λ (Figure 11).

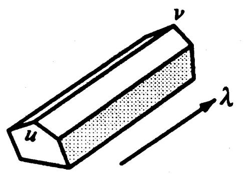


Figure 10

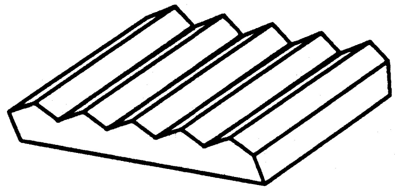


Figure 11

5. Conclusions.

In this paper, we established a tight bound on the number of extrema of a three-dimensional polyhedron in terms of the number r of its reflex edges. We first proved a $2r+2$ upper bound, which we matched by exhibiting a polyhedron that has r reflex edges and $2r+2$ extrema with respect to some direction, for every integer $r \geq 0$.

The open question that immediately comes to mind is what happens in higher dimensions. What is the number of extrema (with respect to some direction) of a d-dimensional polyhedron? What is needed first, however, is a measure of non-convexity of such a polyhedron. How do the notions of reflex vertices of a polygon and reflex edges of a 3-dimensional polyhedron generalize in four and higher dimensions?

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