CONVEX MINIMIZATION ON A GRID AND APPLICATIONS

Extended Abstract

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1 - INTRODUCTION

This paper discusses a discrete version of the convex minimization problem with applications to the efficient computation of proximity measures for couples of convex polyhedrons. Given a d-variate convex function f with bounded domain, we want to find the hyperrectangle, or cell, of an isothetic grid in $\mathbb{R}^d$ which contains the point of minimum. The grid is supposed to be finite but not necessaily regular.

The proposed algorithm solves the problem if we are able to compute *sound halfspaces*, i.e. halfspaces containing the point of minimum, whose interior do not intersect suitably chosen *elementary regions*. Each elementary region is the intersection of a set of isothetic hyperplanes and one grid cell. The number of computations of sound halfspaces is a useful measure of the performance of the algorithm, since their computation may be expensive, depending on how f is defined. A straightforward upper bound is $O(\log^d n)$ for fixed dimension d and a grid based on n intervals in all directions. However, if the grid is almost uniform, in a sense to be made precise later, $O(\log n)$ such computations are sufficient to obtain the answer.

The case $d = 2$ is of particular interest for its possible applications. Moreover, the algorithm can be adapted to require $O(\log n)$ computations of sound halfplanes in the average. This allows, for example, to detect collisions between two convex polyhedrons in three dimensions with about n vertices in $O(\log^2 n)$ average time, after $O(n \log n)$ preprocessing time and using $O(n)$ storage.

Most of the work on convex minimization is related to optimization problems (see for example [Nem83]). The properties of multivariate convex functions are not used very often in computational geometry, but a noticeable example can be found in [Gün91].

The original motivation of this work was designing efficient operations on a geometric modeler for motion planning purposes, [Mir89], given the available drum based polyhedron representation. (A drum is a particular polyhedron such that all the vertices belong to the boundary of two parallel faces, [Cha87], [Dob83]). However, following a completely different approach based on hierarchical representations, several proximity problems can be solved in $O(\log^2 n)$ worst-case time, [Dob90a], [Dob90b]. Depth of collision and other proximity problems with applications to motion planning are also discussed in [Sri93] and [San92], where the proposed algorithms, although considerably less efficient (linear-time), do not rest on the properties of the representation.

Collision or intersection detection algorithms are usually used in planning systems where the desired path is found by iterating a guess-and-modify process, [Cam90], [Can86]. Alternatively, computing the distance of two objects can play a similar role, [Gil90], [Lin91]. Recently a sublinear algorithm has been developed for detecting intersections of convex polyhedrons in more than three dimensions, [Gün91]. However, the preprocessed representations are not invariant by rotation.

The organization of the paper follows. In section 2 we introduce a new general approach to several proximity problems (distance, collision, depth of collision, intersection detection) involving two convex bodies, which motivates the study of the algorithm scheme outlined in section 3. The special case of a bivariate convex function is further discussed in section 4. In section 5 the previous results are applied to detect collision configurations for couples of polyhedrons in three dimensional space. Finally, some experimental results are mentioned in section 6.

2 - MOTIVATION AND STATEMENT OF THE PROBLEM

A basic observation motivating this work is that some proximity measures for two convex bodies can be determined by minimizing a convex function which represents the same measure relative to couples of simpler portions of the bodies, as suggested by the following proposition.

To see this we need some notation. For two convex regions X_1, X_2 and a direction d in $\mathbb{R}^n$, let $\Delta(X_1,X_2)$ be the distance of X_1 and X_2, and $\Delta_d(X_1,X_2)$ the (possibly negative) extent of the translation along d taking X_1 into contact with X_2, if such a translation exists.

Moreover, call o_i the origin and $x_{i1}, ..., x_{in}$ the unit vectors parallel to the axes of a local reference frame for the region X_i. Denote by W_i the matrix $[x_{i(n-k_i+1)} \cdots x_{in}]$, by ζ a vector of dimension k_i, and consider the family of flats (affine sets) $\sigma_i(\zeta) = \{ p \in \mathbb{R}^n \,/\, W_i^T(p-o_i) = \zeta \}$ (for $k_i=1$ a family of hyperplanes perpendicular to x_d and at a distance ζ from o_i).

<u>Proposition</u> 2.1 - Let X_1, X_2 be closed convex regions and d a direction in $\mathbb{R}^n$. Consider the families $\mathcal{S}_i = \{ \sigma_i(\zeta) \,/\, \zeta \in \mathbb{R}^{k_i} \}$, $i = 1, 2$. Then, the following are convex functions with domain in $\mathbb{R}^{k_1+k_2}$:

(a) $\phi(\zeta_1, \zeta_2) = \Delta(X_1 \cap \sigma_1(\zeta_1), X_2 \cap \sigma_2(\zeta_2))$ (b) $\varphi(\zeta_1, \zeta_2) = \Delta_d(X_1 \cap \sigma_1(\zeta_1), X_2 \cap \sigma_2(\zeta_2))$

By definition, $\Delta(X_1,X_2)$ is the ordinary distance, whereas $\Delta_d(X_1,X_2)$ is the collision translation in direction d. Other proximity queries and measures are directly related to $\Delta_d(X_1,X_2)$. In particular, X_1 and X_2 intersect if and only if $\Delta_d(X_1,X_2) \leq 0$ for d the direction of o_2-o_1. Furthermore, if $X_1 \cap X_2 \neq \emptyset$, $-\Delta_d(X_1,X_2)$ is the depth of collision in direction $-d$.

Let us omit the subscrits for a while. If X is a convex polyhedron in $\mathbb{R}^n$, then $X \cap \sigma(\zeta)$ is also a polyhedron in $\mathbb{R}^{n-k}$, whose topology varies at critical points ζ. Such critical points in $\mathbb{R}^k$ correspond to facets of $X \cap \sigma(\zeta)$ which may either appear or disappear and lie on the hyperplanes of an arrangement $\mathcal{A}$ in $\mathbb{R}^k$. An estimate of the complexity of $\mathcal{A}$ is $O(m^{k \min\{ n-k+1, \lfloor n/2 \rfloor \}})$, where m is the number of facets of X.

For instance, consider the case $n = 3$ and $k = 1$. Each planar section $X \cap \sigma(\zeta)$ of the polyhedron X is a polygon whose vertices are determined by some of the edges of X and such edges are different for two sections only if an intermediate planar section contains a vertex of X. So, the critical values of ζ correspond to planes $\sigma(\zeta)$ which contain a vertex of X.

Being concerned with proximity measures relative to two convex polyhedrons X_1 and X_2, we are looking for (ζ_1^*, ζ_2^*) such that $X_1 \cap \sigma_1(\zeta_1^*)$ and $X_2 \cap \sigma_2(\zeta_2^*)$ realize the measure. An interesting question is what cell of the arrangement $\mathcal{A}_i$ contains ζ_i^* (for $i = 1, 2$). In the following we will not discuss the problem in its full generality, but only consider arrangements based on isothetic hyperplanes. In $\mathbb{R}^{k_1+k_2}$ such hyperplanes give rise to an isothetic grid, whose cells are hyperrectangles $\{ (\zeta_1,\zeta_2) \,/\, \zeta_1 \in c_1 \wedge \zeta_2 \in c_2 \}$ for cells $c_1 \in \mathcal{A}_1$ and $c_2 \in \mathcal{A}_2$. Notice however that this is always the case for $k_1 = k_2 = 1$, when the sets of critical values of ζ_1 and ζ_2 determine a rectangular grid in $\mathbb{R}^2$. Such critical values for X_i are as many as the number of vertices of X_i.

Here is the reference problem (where $d \geq 1$ and the ζ_i's are normalized into the interval $[0,1]$):

> Given a convex function f with domain in $[0,1]^d$ and d sets of real constants $\{ \zeta_{ij} \,/\, j = 0,...,h_i \}$ such that $\zeta_{i0} = 0$, $\zeta_{ih_i} = 1$, $\zeta_{ij} < \zeta_{ij+1}$ for $0 \leq j < h_i$, find the grid cell $[\zeta_{1j_1}, \zeta_{1j_1+1}] \times ... \times [\zeta_{dj_d}, \zeta_{dj_d+1}]$ which contains the point of minimum of f.

3 - ALGORITHM SCHEME

Call *sound* a closed halfspace in $\mathbb{R}^d$ containing the point of minimum of the convex function f. Moreover, call *elementary* a region in $\mathbb{R}^d$ that results from the intersection of a set of t isothetic hyperplanes, $1 \le t \le d$, and one grid cell. In particular all the points in $[0,1]^d$ are elementary regions. We assume we are able to solve any instance of the following problem: Given an elementary region e, build a sound halfspace $H(e,f)$ without points of e in its interior.

Under this assumption, a simple algorithm solves the reference problem by computing sound halfspaces a polylogarithmic number of times. Let $h_i \le h$ for $i = 1, \ldots, d$.

Proposition 3.1 - For fixed dimension d, $O(\log^d h)$ computations of sound halfspaces are sufficient to find the grid cell containing the point of minimum.

In most cases, the number of computations of sound halfspaces can be drastically reduced by exploiting standard techniques such as the method of centers of gravity and the ellipsoid method (MCG and MMCG, respectively, in [Nem83]). Basically, these techniques rest on the fact that we can repeatedly shrink the volume of a convex region $G \subseteq \mathbb{R}^d$ containing the point of minimum by cutting it by hyperplanes perpenicular to the gradient of the convex function f at the centroid of G. If G is an ellipsoid, it is updated after a cut by taking the smallest ellipsoid containing the appropriate portion, and the volume is reduced by a costant factor $\alpha_d = (d/\sqrt{d^2-1})^d \sqrt{(d-1)/(d+1)}$ for $d > 1$, or $\alpha_d = 1/2$ if $d = 1$, [Nem83].

Below, the refined algorithm is outlined. Intuitively, the ellipsoid method is applied in its usual form (steps 1-3, with $A = e = \{c\}$) until the width of the ellipsoid E in some direction is comparable with the width of the intersected grid cells near the center c. When such condition holds ($dim(A) > 0$), sound halfspaces $H(e_j,f)$ are computed for a limited number of elementary regions e_j, adjacent in the affine subspace A.

In a subspace of maximal dimension perpendicular to A, the projection of E is still an ellipsoid E' whose centroid c' is the projection of A. Moreover, the projection of $\cap_j H(e_j,f) \cap E$ falls into a sound halfspace (relative to the subspace) without c' in its interior. In other words, the ellipsoid method is applied in the projection subspace (steps 1-3, with $dim(A) > 0$).

During the above process, the ellipsoid E becomes thin in more and more directions. Eventually, the dimension of A will be $d-1$ and E' an interval in the projection subspace of dimension 1. As a result of halving E', the whole isothetic region R, which keeps the candidate grid cells, can be halved in the corresponding direction, the cutting hyperplane being A itself (step 4).

Algorithm *GridMin* — input: a d-variate convex function f, a grid g partitioning the unit hypercube
output: the grid cell containing the point of minimum of f

R := unit hypercube, partitioned according to the grid g;
E := sphere circumscribing the unit hypercube;

while R contains more than one grid cell
do begin

1 c := centroid of the ellipsoid E;

2 let A be the isothetic flat of maximal dimension
such that $c \in A$ and all the elementary regions $e \subseteq A$
contained in R and intesecting E share one grid vertex;

3 compute $H(e,f)$ for every such e and update E;

4 **if** the dimension of A is $d-1$
then update R w.r.t. the hyperplane A

end;

output R

Define coefficients $\rho_i = h_i \cdot min\{ \zeta_{ij+1} - \zeta_{ij} / 0 \leq j < h_i \}$. By definition, $0 < \rho_i \leq 1$ and $\rho_i = 1$ only if the corresponding grid lines are equally spaced. Let $h_i \leq h$ and $\rho_i \geq \rho$ for $i = 1, ..., d$. Then:

Proposition 3.2 - For fixed dimension d, the grid cell containing the point of minimum can be found after $O(\log h + \log(1/\rho))$ computations of sound halfspaces.

However, even if the computation of a sound halfspace requires constant time, the overall cost is $O((\log h + \log(1/\rho)) \log h)$. Finally, fix a constant w and define *almost uniform* the grids such that $\rho \geq h^{-w}$. Considering only almost uniform grids, even for small w, is not a serious restriction from a practical viewpoint because of the limited prcision of floating point arithmetic. We have:

Proposition 3.3 - For fixed dimension d, given an almost uniform grid, the cell containing the point of minimum can be found after $O(\log h)$ computations of sound halfspaces.

A crucial role in the worst case analysis of algorithm *GridMin* is played by the distance between the closest grid hyperplanes. One may try to modify the algorithm in such a way that the computations of sound halfspaces are $O(\log(1/\delta))$, where δ is the shortest distance of the point of minimum from a grid hyperplane. We will see how this can be accomplished at a reasonable cost in the case $d = 2$. However, the following more general result can be proved:

Proposition 3.4 - Let be given any grid of fixed dimension d and assume that: (i) the point of minimum of the convex function f is a random variable with uniform distribution in the unit hypercube; (ii) the cost of locating the point of minimum in terms of computations of sound halfspaces is $T(\delta) = C \log(1/\delta)$ for δ the distance between such a point and the closest grid hyperplane. Then the expected computational cost is $O(\log h)$.

4 - THE CASE OF PLANAR GRIDS

The case $d = 2$ is worth receiving particular attention for its possible applications. According to the interpretation suggested by proposition 2.1, both the distance and the collision translation relative to two convex bodies in any space can be reduced to analogous subproblems for couples of hyperplanar sections and minimization of a convex function of arity two.

When $d = 2$ algorithm *GridMin* can be modified to require $O(\log h)$ computations of sound halfplanes in the average. For proposition 3.4, it is sufficient to prove that the solution can be found after $O(\log(1/\delta))$ computations of sound halfplanes if δ is the distance between the point of minimum and the closest grid line. However, it is important to achieve the goal at low additional cost (i.e. $O(\log h)$ time per iteration, since otherwise the overall performance would deteriorate).

Roughly speaking, at each iteration, a sound halfplane through the centroid is computed and the ellispe E updated in the standard way. Then, two vertical (horizontal) straight lines closest to the vertical (horizontal) median grid line from opposite sides and intersecting E within an elementary region are looked for to reduce R (the medians are relative to R). All these operations require $O(\log h)$ time, provided the grid intervals are represented by interval trees with intermediate nodes storing the widths of the largest intervals in the left and right subtrees. Finally we have:

Proposition 4.1 - Given any planar grid, the rectangle containing the point of minimum can be found after $O(\log h)$ computations of sound halfplanes in the average. Moreover, all the other operations require $O(\log^2 h)$ time in the average.

5 - APPLICATION TO COLLISION DETECTION

In this section we are concerned with the following problem in $\mathbb{R}^3$: given two convex polyhedrons P and Q, find out the collision configuration for P moving in a ray r. To apply the algorithm sketched in the previous section, we must be able to compute sound halfplanes for any elementary region. In $\mathbb{R}^2$ there are two kinds of elementary regions, i.e. points, representing couples of planar sections of the polyhedrons, and particular segments, representing couples section-drum.

So, our method rests on the solution of these subproblems in *three* dimensional space:
(a) Given two convex polygons P and Q, find the collision configuration for P moving in a ray r.
(b) Given a convex polygon P and a drum D, find the collision configuration for P moving in a ray r.
According to our interpretation of the grid and the convex function, once the grid rectangle containing the point of minimum has been detected by *GridMin*, we know the two drums, one for each polyhedron, that are involved in the collision. Thus, there only remains to solve the problem:
(c) given two drums D and E, find the collision configuration for D moving in a ray r.

Given a point, independently of whether it belongs to the domain of the convex function or not, i.e. independently of whether the corresponding planar sections collide or not, a sound halfplane can be constructed. In the latter case we refer to suitable *separation* configurations. Similar considerations hold when the elementary regions are segments. A list of some results from [Mir94] follows.

Proposition 5.1 - Given either a contact or separation configuration of two planar sections (a planar section and a drum), a sound halfplane can be built in $O(\log n)$ for polyhedrons with $O(n)$ vertices.

Proposition 5.2 - A collision configuration for an m-sided and an n-sided polygon in three dimensional space can be computed in $O(\log m + \log n)$ time. Moreover, if the polygons do not collide, a suitable separation configuration is built at the same cost. (Algorithm *PgPgColl*).

Proposition 5.3 - A collision configuration for an m-sided polygon and an n-faceted drum in three dimensional space can be computed in $O(\log m + \log n)$ time. Moreover, if there is no collision, a suitable separation configuration is built at the same cost. (Algorithm *PgDrColl*).

Proposition 5.4 - A collision configuration for two drums having n faces in total can be computed in $O(\log^2 n)$ time. (Algorithm *DrDrColl*).

To conclude this section, we analyse the overall computational costs. We assume drum based representations of polyhedrons with $O(n)$ vertices, whose storage and preprocessing time (starting from incidence graphs) can be reduced to $O(n)$ by means of Layered DAGs, [Ede86]. The interval trees require $O(n)$ storage too, and can be built in $O(n \log n)$ from the (unordered) set of vertices of each polyhedron. (There are as many grid lines as the number of vertices of the polyhedrons).
At the generic iteration of algorithm *GridMin* at most a constant number of sound halfplanes are computed at a cost $O(\log n)$ by using algorithms *PgPgColl* and *PgDrColl*. Moreover, all the other operations require $O(\log n)$ time per iteration. Since the number of steps is $O(\log n)$ in the average, and the final application of algorithm *DrDrColl* costs $O(\log^2 n)$, we have proved the following result:

Proposition 5.5 - A collision configuration for two polyhedrons having $O(n)$ vertices can be computed in $O(\log^2 n)$ time in the average.

6 - SOME EXPERIMENTAL RESULTS

For $d = 2$, at each iteration of *GridMin*, the area of the ellipse E is reduced by a factor $\alpha_2 = 4/(3\sqrt{3}) \cong 0.77$. On the other hand, the area of each subregion that results from splitting any convex region G by a straight line through its centroid is at most $5/9 \cong 0.56$ the area of G. This suggests a polygonal representation of the region containing the point of minimum and a refinement of *GridMin* based on this observation has been experimented.

The experimens were planned to estimate the actual influence of ρ_1, ρ_2, and of other qualitative factors, such as the shape of the convex function and the density of grid lines in the vicinity of the point of minimum. Every row in the table summarizes 1,200 tests, giving the average and maximum number of iterations of the main loop. The fourth column shows the maximum number of sides of the polygon G ever found (at any step of any of the 1,200 computations), but it should be noticed that the actual number of sides is only occasionally greater than 8 and in most of the computations that value is never exceeded. The last two columns give further information rlative to the possible application of algorithm *GridMin* to collision detection, i.e. the average number of required applications of the algorithms *PgPgColl* and *PgDrColl*. (A finer analysis can be found in [Mir94]).
It is interesting to compare these results with the analogous measures of performance when an ellipse is used to bound the region which contains the point of minimum. In this case, for another set of 3,600 tests planned similarly, the second column gives 38.02, 53.15 and 66.55, respectively, and the

third 54, 80 and 105. From such results, we should definitely prefer the use of a polygon to the use of an ellipse in practical situations.

grid lines h_1, h_2	average num. of iterations	max. number of iterations	max number of sides of G	applications of *PgPgColl*	applications of *PgDrColl*
10^3	18.03	28	14	33.06	5.90
10^4	23.61	39	18	42.63	7.09
10^5	28.59	38	16	50.96	8.60

7 - CONCLUSIONS

Of course, an algorithm solving the reference problem may benefit of more than one of the discussed techniques and merge different features. Nevertheless, the experimental results reinforce the intuitive expectation that the local density of the grid does not have much influence on the average performances and seem to suggest that a simple implementation could be preferable.

As said before, the original motivation of the work discussed in this paper was the design of efficient operations for motion planning. Later, some generalizations of the involved problems have been studied. However there remain several open problems:
- Is it possible to refine algorithm *GridMin*, at least for $d = 2$, in such a way that $O(\log h)$ computations of sound halfspaces are sufficient even in the worst case? and in that case, is it possible to avoid paying some additional costs in terms of other needed operations?
- To what extent can the refinement described in section 4 be generalized to more dimensions?
- What happens if the isothetic grids are replaced by more general arrangements, as mentioned in section 3?
- Is it possible to efficiently solve some proximity problems for convex polyhedrons in spaces of more than three dimensions by recursively applying the techniques introduced here, relative to a bivariate convex function?

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