Edge Guards on A Fortress
(Extended Abstract)

S.M. Yiu and A. Choi
Department of Computer Science
University of Hong Kong
Pokfulam Road, Hong Kong
{syiu, choi}@csd.hku.hk

Abstract

The *fortress* problem was originally posed to determine the number of vertex guards sufficient to cover the exterior of a polygon with n vertices. O'Rourke and Wood [J. O'Rourke, *Art Gallery Theorems and Algorithms*, Oxford University Press, 1987] showed that $\lceil n/2 \rceil$ vertex guards are sometimes necessary and always sufficient. In this article, we consider a variation of the problem which allows a guard to patrol an edge of the polygon instead of standing stationarily. Tight bounds of $\lceil n/3 \rceil$ and $\lfloor n/4 \rfloor + 1$ are obtained for general polygons and rectilinear polygons, respectively.

1 Introduction

The fortress problem is a variation of the *art gallery problem*. Refer to [4, 5] for an excellent description of the latter. This paper deals with the following problem: Given a simple polygon with n vertices, select a set of edges such that every point in the exterior of the polygon is *covered* by at least one of the selected edges. An exterior point x is said to be covered by an edge AB if there exists a point y on AB such that xy does not intersect the interior of the polygon. In the real-world analogy of the problem, each selected edge of a polygonal fortress can be thought of as being patrolled by a guard (called an *edge guard*) to protect the fortress against invasion, and thus the problem's name. In section 2, we will show that there exist polygons requiring $\lceil n/3 \rceil$ edge guards and the bound will be proved to be tight. Rectilinear polygons are considered in section 3. $\lfloor n/4 \rfloor + 1$ edge guards are sometimes necessary and always sufficient in this case.

2 General Polygons

Given a polygon, each connected region inside its convex hull but exterior to the polygon is called a *pocket*. For example in figure 1, p_1, p_2, and p_3 are pockets where p_1 has 2 edges and p_3 has 4 edges. A *triangulation graph* of a pocket is a graph whose embedding is a triangulation of the pocket. Its vertices are the vertices of the pocket and its edges are the edges of the pocket plus the diagonals of the triangulation. A triangulation graph is *dominated* by a set of edges (guards) if every triangular face of the graph shares a vertex with at least one of the guards.

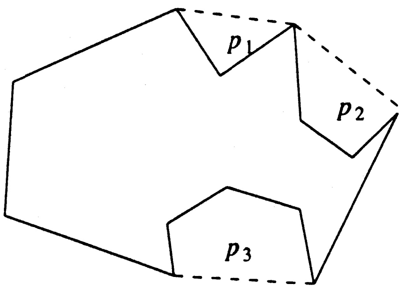


Figure 1

Lemma 1: If the polygon is convex or all of its pockets have less than four edges, then $\lceil n/3 \rceil$ edge guards suffice to cover the exterior of the polygon. Also the guards dominate any triangulation graph of the pockets.

Proof: Lemma 1 follows by placing the guards on every third edge. It can also be shown that visibility is only required at the end points of each edge guard.

Lemma 2 [4]: After the pockets are triangulated, among all pockets of more than 3 edges, there exists a diagonal d which cuts off a region with exactly 4, 5, or 6 edges.

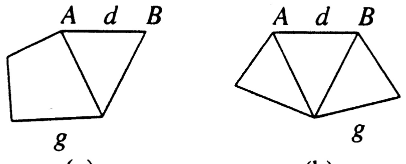
Proof: Omitted [4].

Theorem 1: $\lceil n/3 \rceil$ edge guards are sometimes necessary and always sufficient to cover the exterior of a polygon with n vertices and the guards can be chosen to dominate all triangulation graphs of the pockets. Visibility is required only at the end points of each edge guard.

The necessity can be shown by a convex polygon.

The sufficiency will be proved by induction. Lemma 1 establishes the induction basis. If the polygon has no pocket with more than 3 edges, lemma 1 applies. Let d be the diagonal in Lemma 2, three cases need to be considered in covering the region cut off.

Case 1: d cuts off 4 edges.
The region cut off from the pocket can be covered and dominated by two edge guards with end points at A and B or one edge guard at position g in figures 2a and 2b.



(a) (b)

Figure 2

Having replaced the four edges by the diagonal d, the polygon will have $(n-3)$ edges. By the hypothesis, $\lceil n/3 \rceil - 1$ edge guards are sufficient to cover the exterior and dominate all triangulation graphs of the pockets. However, if a guard is placed at d when applying the induction hypotheses, it can always be replaced by two edge guards with end points at A and B. Since the construction only requires visibility at the end points of the guard edges, the result follows.

Case 2: d cuts off 5 edges.
Similar to Case 1.

Case 3: d cuts off 6 edges.
There are four configurations in which the region that is cut off can be triangulated; figure 3 shows one of them. Instead of cutting through d, cut along diagonals AC and BC. The resulting polygon will have $(n-4)$ edges. By hypothesis, $\lceil n/3 \rceil - 1$ guards suffice. Also, by hypothesis, triangle ABC is dominated and at least one of the vertices of ABC will be covered. For example, if a guard is placed at BC, replace it by BD and CE. The original polygon will require a total of $\lceil n/3 \rceil$ guards. All other cases are similar [8].

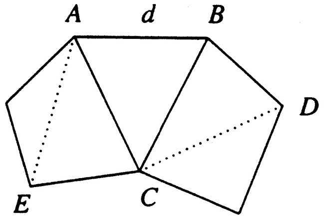
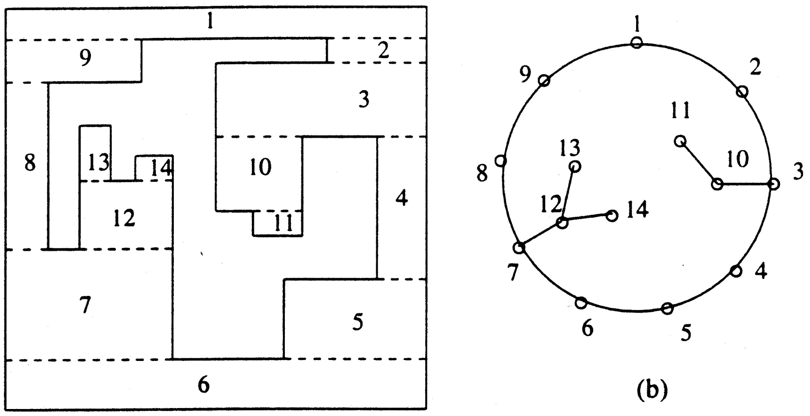


Figure 3

3 Rectilinear Polygons

To show that $\lfloor n/4 \rfloor + 1$ edge guards are sufficient for a rectilinear polygon, it is enclosed in a rectangle and its horizontal edges are extended from its convex vertices to partition the region between it and the rectangle as shown in figure 4a. This partitioning method is the same as the *rectangular decomposition* used in [3]. The region is decomposed into rectangles. To cover the exterior of the polygon is equivalent to covering all resulting rectangles since the enclosing rectangle can be arbitrarily large. Assume the polygon is in general position (that is, no two nonadjacent convex vertices have the same y coordinates), the total number of such rectangles will be $n/2 + 2$. We can assume the given rectilinear polygon is in general position without loss of generality (see [3, 4]).

This set of rectangles has the following properties (see also [3]). Two rectangles are *adjacent* if they share a horizontal chord. Any two adjacent rectangles can be covered by a vertex guard (or an edge guard). The dual of the partition (the graph in which each rectangle is represented by a node and two nodes are connected if the corresponding rectangles share a horizontal chord, see figure 4b and [3]) is simply a cycle with attached trees. If all the attached trees have only one node, we say that the graph is in *reduced configuration* (cf. reduced triangulation in section 5.2 of [4]). Nodes on the cycle are called *cycle nodes*; all other nodes are called *tree nodes*. A cycle node which has a tree attached to it is called a *root* node. A node such as node 12 is called the parent of nodes 13 and 14. The topmost (or bottommost) edge of the polygon can cover at least three rectangles.



(a) (b)

Figure 4

To prove the main result of this section, examples of a family of polygons will be given to establish the necessary number of guards. For sufficiency, we first show that for any rectilinear polygon whose partition is in reduced configuration, $\lfloor n/4 \rfloor + 1$ guards suffice to cover the exterior. Then it will be shown that a rectilinear polygon whose partition is in general configuration can be reduced to one in reduced configuration while maintaining the correct number of guards required.

Theorem 2: $\lfloor n/4 \rfloor + 1$ edge guards are sometimes necessary and always sufficient to cover the exterior of a rectilinear polygon with n vertices.

The necessity is demonstrated by the following figures. In figure 5a, a polygon with 12 vertices requires 4 edge guards. In figure 5b, an addition of 4 vertices necessitate one additional guard. Figure 5b can be generalized to polygons with number of edges equal to $4k$ for all values of $k > 2$. The number of guards necessary, $\lfloor n/4 \rfloor + 1$, can be easily established.

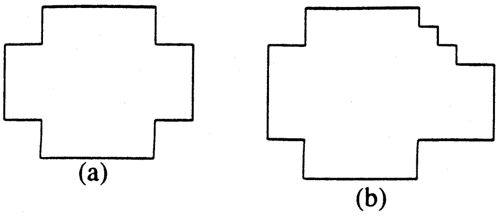


Figure 5

Lemmas 3 and 4 follow from the fact that any two adjacent rectangles must share a vertical edge because the polygon is in general position (for a similar argument, refer to [3]).

Lemma 3: Each root node is adjacent to at most two tree nodes.

Proof: Omitted [8].

Lemma 4: If a root node is adjacent to two tree nodes, one guard is sufficient to cover all three corresponding rectangles. If a root node is adjacent to exactly one tree node, one guard suffices to cover the two corresponding rectangles and can in addition cover the rectangle corresponding to either of the two adjacent cycle nodes.

Proof: The proof enumerates the possible positions of the guard g as shown in figure 6. Figure 6a shows the case when the root node (A) is adjacent to two tree nodes (B and C) and two cycle nodes (D and E) while figures 6b and 6c show some of the cases when the root node is adjacent to only one tree node (B). All other cases are handled similarily [8].

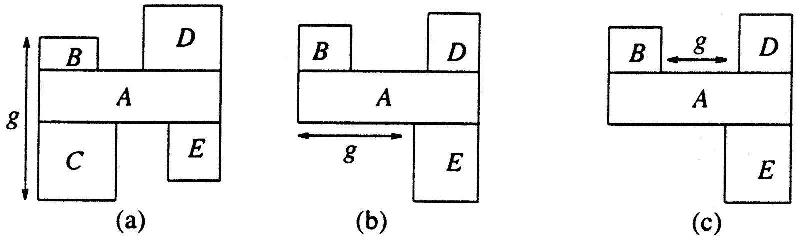


Figure 6

Lemma 5: To cover the exterior of any rectilinear polygon whose partition is in reduced configuration, $\lfloor n/4 \rfloor + 1$ edge guards are sufficient.

Proof: The proof assigns guards to cover the rectangles while maintaining that the average number of rectangles covered by each guard is at least two. All root nodes are grouped with their own tree nodes if the latter exist. By lemma 4, one guard can be assigned to cover each group of rectangles. Then, starting from any root node, consider the cycle nodes between two consecutive root nodes in clockwise order. If there are an even number of cycle nodes, each two of them can be grouped. Otherwise, depending on the number of attached tree nodes of the first root node, choose a different method to group the nodes as follows.

If the root node is adjacent to only one tree node, include the next cycle node into the group of this root node as shown in figure 7. By lemma 4, this group can be covered by a single edge guard.

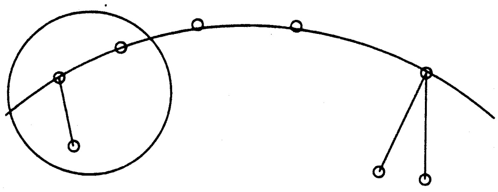


Figure 7

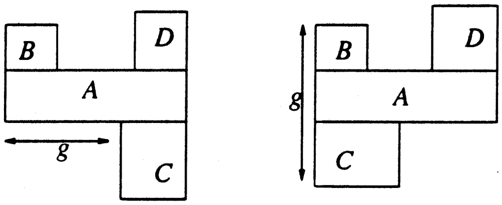
If the root node is adjacent to two tree nodes, group the next cycle node in its own group. On average, each guard will still cover at least two rectangles.

After this procedure, an even number of cycle nodes will remain between two consecutive root nodes. Group these in groups of two. Since rectangles in the same group can be covered by one guard, each guard is responsible for at least two rectangles. If the number of rectangles is odd, there must exist an extra group of three rectangles which is not paired with a group of one rectangle, so the total number of guards required is $\lfloor (n/2 + 2)/2 \rfloor$ which is equal to $\lfloor n/4 \rfloor + 1$. There is a special case in which no root node exists. Since the topmost edge can cover three rectangles, at most $\lfloor n/4 \rfloor + 1$ guards suffice. The result follows.

Lemma 6: Every rectilinear polygon whose partition is in general configuration can be changed to one in reduced configuration, while maintaining the invariant that every assigned guard covers at least two rectangles.

Proof: In a manner similar to a root node, a tree node can be adjacent to at most four nodes of which one is a parent node. To reduce the polygon, the following procedure is repeated. Starting at the leaves, a leaf is grouped with its parent if the parent is of degree 2. The group is removed. Continue until we cannot proceed further. At this point, either the tree is reduced or a parent has degree more than two. The rectangles are then grouped as follows depending on the degree of the parent. Let A be the parent of the leaf or leaves. Assume A is not a root node, otherwise the tree is reduced.

(a) If A is of degree three, group A with the two attached leaves as shown in figure 8 (only two configurations are shown) and remove the whole group. One guard suffices to cover all removed rectangles. In the figures, B and C are leaves while D is the parent of A. The guard is placed at g to cover the removed part. All other configurations are similar [8].

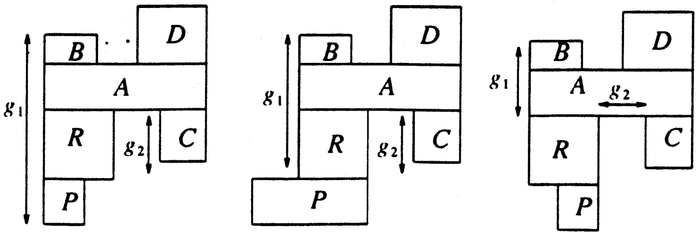


(a) (b)

Figure 8

However, in figure 8a, the position of g may not be an edge of the original polygon because of previously removed rectangles. In this case, we must reshuffle the grouping of A, B, and C together with rectangles originally connected to the position of g in the original polygon. Let R denote the rectangle connected immediately to A originally (figure 9). R will have degree less than four as can be easily verified. In fact, no two nodes with degree four can be next to each other. So, two subcases have to be considered depending on the number of rectangles in the group of R.

If the number of rectangles is 2 where R and P are in the same group, there are three subcases. In figures 9a and 9b, regroup the rectangles into two groups, A, B, R, and P as one group and C as another. In figure 9c, R, P, and C will be one group while A and B will be another. The guards for the two groups can be placed at the positions g_1 and g_2 in each case.



(a) (b) (c)

Figure 9

If the number of rectangles is 3, group the rectangles as shown figure 10. *B*, *A*, *R* and *P* will be in the same group while *Q* and *C* will be another. The required guards can be placed at the positions g_1 and g_2. In each of the above cases, two groups require two guards and a total of at least five rectangles are covered, so the invariant of the procedure is still maintained.

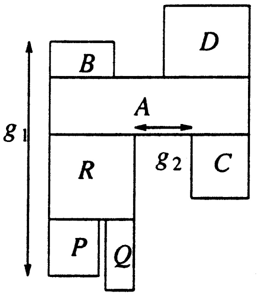


Figure 10

(b) If *A* is of degree four, use a similar technique as in (a), we can group the rectangles in two groups with four rectangles covered [8].

The whole procedure is repeated until all trees are reduced. Then lemma 5 is applied to the resulting polygon. If lemma 5 requires that a guard be placed at a position which is not an edge of the original polygon (only possible in figure 6b), the grouping can be reshuffled as in figure 9. This completes the proof of theorem 2.

4 Related Problems

In this paper, two results concerning edge guards were obtained: to cover the exterior of a general polygon, $\lceil n/3 \rceil$ edge guards are sometimes necessary and sufficient, while for a rectilinear polygon, $\lfloor n/4 \rfloor + 1$ edge guards suffice and the bound is tight.

Most other results for edge guards solved the problem of covering the interior of a polygon instead of the exterior. These results include [1], [2], and [7]. These papers investigated the number of edge guards required to cover the interior of polygons with specific shapes such as monotone, rectilinear monotone, and spiral polygons. Several tight bounds were obtained. However, for general rectilinear polygons and simple polygons, tight bounds of the number of edge guards are not known until now [4, 5]. Other related open problems include finding a minimum number of edge guards sufficient for any star-shaped polygons. In [6], it was shown that $\lfloor n/5 \rfloor$ edge guards are not sufficient.

References:

[1] Iliana Bjorling-Sachs and Diana L. Souvaine, *A Tight Bound for Edge Guards in Monotone Polygons*, DIMACS Technical Report 92-52, Rutgers University, 1992.

[2] Iliana Bjorling-Sachs, *A Tight Bound for Edge Guards in Rectilinear Monotone Polygons*, DIMACS Technical Report 93-12, Rutgers University, 1993.

[3] E. Gyori et al, *Generalized Guarding and Partitioning for Rectilinear Polygons*, Technical Report CMPT94-01, School of Computing Science, Simon Fraser University, January 1994.

[4] J. O'Rourke, *Art Gallery Theorems and Algorithms*, Oxford University Press, 1987.

[5] Thomas C. Shermer, *Recent Results in Art Galleries*, Proceedings of the IEEE Vol 80 No 9, 1992, p.1384-1399.

[6] R.V. Subramaniyam and A.A. Diwan, *A Counterexample for the Sufficiency of Edge Guards in Star Polygons*, Information Processing Letters 40, 1991, p.97-99.

[7] S. Viswanathan, *The Edge Guard Problem for Spiral Polygons*, Proc. of the 5th Canadian Conf. on Comput. Geom., 1993, p.103-108.

[8] S.M. Yiu and A. Choi, *Edge Guards on A Fortress*, Technical Report 94-06, Department of Computer Science, University of Hong Kong, 1994.