Edge guarding a triangulated polyhedral terrain

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Abstract: In this note we show that $\lfloor n/3 \rfloor$ guards are always sufficient to guard a triangulated polyhedral terrain on n vertices. This is equivalent to showing that $\lfloor n/3 \rfloor$ edges are sufficient to cover all of the faces of a planar triangulation on n vertices.

1 Introduction

The art gallery problem, originally posed in 1973, is to determine the minimum number of guards sufficient to cover the interior of a simple n-sided polygon. In 1975 Chvatal resolved the problem showing that $\lfloor n/3 \rfloor$ guards are always sufficient. Since this time many variants of the art gallery problem have been studied [O87]. Here we consider the variant in which the guards are permitted to patrol along the edges of the polyhedral terrain they wish to guard.

A *polyhedral terrain* is a polyhedral surface in three dimensions such that the intersection of the terrain with a vertical line is either empty or a point. A polyhedral terrain is *triangulated* if each of its faces is a triangle. Two points x and y of a terrain are said to be *visible* if the line segment $\overline{xy}$ does not contain any points below the terrain. A point x of a terrain is said to be *visible to an edge e* if there exists a point y on e such that x and y are visible. A set of edges is said to *guard* a terrain if every point of the terrain is visible from one of the edges. We call the problem of finding such a set of edges the *terrain edge guarding problem*. It has been shown in [BSTZ92] that $\lfloor (4n-4)/13 \rfloor$ edges are sometimes necessary to guard a terrain. It is the purpose of this note to establish that $\lfloor n/3 \rfloor$ edges are always sufficient.

Let $G=(V,E)$ be a planar triangulated graph on n vertices. A set of edges H in G is said to guard G if every face of G contains at least one vertex in the vertex set of H. We call the problem of finding such a set of edges the *combinatorial edge guarding problem*. It is easy to see that a solution to the combinatorial edge guarding problem is also a solution to the terrain edge guarding problem: associate to a given a terrain T a planar triangulated graph $G(T)$ corresponding to the projections of the vertices and edges of T onto a horizontal plane lying below T. In this note we show that $\lfloor n/3 \rfloor$ edges are always sufficient to guard a planar triangulated graph on n vertices and the result for terrains follows.

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We need the following definitions. A *coloring* of a graph is an assignment of colors to vertices such that no two adjacent vertices receive the same color. The Four-color Theorem states that any planar graph can be colored by at most four colors [AH77]. A *matching* is a subset M of the edges of a graph such that no vertex is contained in more than one edge of M. A matching M is called *maximal* if no other edge can be added to M such that it remains a matching. The size of a matching is the number of edges in it. We note here that given a planar graph, a four-coloring can be found in time $O(n^2)$ [AH77] and a maximal matching in linear time using a greedy algorithm.

2 Main Theorem

Theorem: Every planar triangulation G on n vertices can be guarded with $\lfloor n/3 \rfloor$ edges.

Proof: Let $\{c_1, c_2, c_3, c_4\}$ be the set of four colors used in a coloring of G and let v_1, v_2, v_3 and v_4 be the sets of vertices colored by c_1, c_2, c_3, and c_4 respectively. Notice that since G is triangulated, each face contains three vertices colored by three distinct colors and consequently, any pair of color classes $\{v_i, v_j\}$, $1 \le i < j \le 4$, contains a vertex from each face. Let G_{ij} be the subgraph of G induced by v_i and v_j and let M_{ij} be a maximal matching in G_{ij}, $1 \le i < j \le 4$. An edge guarding of G can be construted by taking the edges of M_{ij} plus one edge incident to each vertex in $v_i \cup v_j$ that is not in any edge of M_{ij}. The size of this edge guarding is given by $|v_i| + |v_j| - |M_{ij}|$. The average size of $|v_i| + |v_j| - |M_{ij}|$ over all i and j is $(3n - \sum_{1 \le i < j \le 4} M_{ij})/6$. Thus, if $\sum_{1 \le i < j \le 4} M_{ij} \ge n$, then at least one of these edge guardings has size less than $\lfloor n/3 \rfloor$ and we are done; so suppose this is not the case.

Consider the sets $M_{12} \cup M_{34}$, $M_{14} \cup M_{23}$, and $M_{13} \cup M_{24}$. We claim that these sets also constitute edge-guardings. We show this for the set $M_{12} \cup M_{34}$, the argument for the other sets is similar. Suppose there is a face f that contains no vertex in the vertex set of $M_{12} \cup M_{34}$. Since each face is colored by three distinct colors, f must contain either an edge whose vertices are colored by c_1 and c_2 or an edge whose vertices are colored by c_3 and c_4; assume the former, the argument for the other case is similar. If this edge is not included in M_{12} then, since the matching is maximal, at least one of the vertices of this edge must be in some edge of M_{12}. But this is a contradiction since we suppose that f contains no vertex in the vertex set of $M_{12} \cup M_{34}$. The average size of the sets $M_{12} \cup M_{34}$, $M_{14} \cup M_{23}$, and $M_{13} \cup M_{24}$ is $\sum_{1 \le i < j \le 4} M_{ij}/3$. Since from the above we have that $\sum_{1 \le i < j \le 4} M_{ij} < n$, at least one of $M_{12} \cup M_{34}$, $M_{14} \cup M_{23}$, and $M_{13} \cup M_{24}$ has size less than $\lfloor n/3 \rfloor$ which completes the proof.

3 Open Problems

A polynomial time algorithm for finding $\lfloor n/3 \rfloor$ edge guards for a triangulated planar graph (or a polyhedral terrain) follows easily from the proof. Since this algorithm involves four coloring the graph it is not very practical. It would be interesting to find a fast algorithm to solve this problem. Also, there remains a small gap between the upper and lower bounds.

References

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