Graph-Theoretical Conditions for Inscribability and Delaunay Realizability

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ABSTRACT

We present new graph-theoretical conditions for inscribable polyhedra and Delaunay triangulations. We establish several sufficient conditions of the following general form: if a polyhedron has a sufficiently rich collection of Hamiltonian subgraphs, then it is inscribable. These results have several consequences:

- All 4-connected polyhedra are inscribable.
- All triangulations without chords or nonfacial triangles are realizable as (combinatorially equivalent) Delaunay triangulations.
- All simplicial polyhedra in which all vertex degrees are between 4 and 6, inclusive, are inscribable.
- All simplicial polyhedra in which all vertex degrees are between 5 and 6, inclusive, are circumscribable.

We also prove stronger results than were previously known concerning matchings in inscribable polyhedra. Specifically, any nonbipartite inscribable polyhedron has a perfect matching containing any specified edge, and any bipartite inscribable polyhedron has a perfect matching containing any two specified disjoint edges. These results are best possible.

1 Introduction

One of the central themes of computational geometry is the study and application of proximity graphs, graphs defined by connecting points that are "close together" in some suitable sense. In the past few years, there has been significant interest in studying the graph-theoretical properties of these objects. For example, Patterson and Yao [28], and Eppstein [17], have studied the diameter of nearest-neighbor graphs. Monma and Suri have characterized the graphs that can be realized as planar minimum spanning trees [26]. Cimikowski has derived necessary conditions for graphs to have realizations as Relative Neighborhood Graphs (RNG's) and Gabriel Graphs [7]. Lubiw and Sleumer have given a sufficient condition for a planar graph to have a realization as a RNG [25]. Bose *et al.* have characterized the trees that can be realized as proximity graphs for several different definitions of proximity graphs [3].

Delaunay triangulations are a particularly useful class of proximity graphs. Closely related to Delaunay triangulations is the class of inscribable graphs. (An exact formulation of the relation is given in Lemma 2.2, below.) The problem of providing a graph-theoretical characterization of inscribable graphs is a long-standing open problem, dating back to René Descartes [19] and formally posed by Jakob Steiner [35]. There has been considerable recent progress on the problem. Jucovič and Ševec have established necessary and sufficient conditions for inscribability in the special case of quadrangular polyhedra satisfying certain constraints on their edge types [24]. Dillencourt has shown that all Delaunay triangulations are 1-tough and have perfect matchings [12], and that any outerplanar triangulation is realizable as a Delaunay triangulation [11]. Rivin has provided a *numerical* characterization of inscribable polyhedra as those polyhedra that admit a certain type of weighting (Characterization 2.1, below). Dillencourt and Smith have provided a graph-theoretical characterization of *trivalent* inscribable polyhedra, and a linear-time algorithm for recognizing them [14]. Nevertheless, a general graph-theoretical characterization has remained elusive. Examples given in [14] illustrate some of the subtleties involved.

In the present paper, we establish graph-theoretical conditions for inscribability and Delaunay realizability that considerably narrow the gap between the most general sufficient conditions and the strongest necessary conditions. In Section 3 of this paper, we establish several sufficient conditions. In particular, we show that any 4-connected planar graph is inscribable (Theorem 3.3), that any 4-connected planar graph can be realized as a combinatorially equivalent Delaunay tessellation (Theorem 3.5), and that any triangulation without chords or nonfacial triangles can be realized as a combinatorially equivalent Delaunay triangulation (Theorem 3.6). Theorem 3.6 provides a partial converse to the results of [12], which imply that a Delaunay triangulation cannot have too many chords or nonfacial triangles. Theorems 3.3 and 3.6 are consequences of a more general result, Theorem 3.1 which says, roughly, that if a planar graph has a sufficiently rich collection of Hamiltonian subgraphs, then it is inscribable. We also establish sufficient conditions for inscribability and circumscribability based solely on vertex degrees (Theorems 3.9 and 3.10). These results, in turn, imply a sufficient condition for Delaunay realizability formulated in terms of vertex degrees and the valence of the outer face (Theorem 3.11).

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In Section 4, we present several necessary conditions for inscribability. In particular, we show that a nonbipartite inscribable polyhedron has a perfect matching containing any given edge, and a bipartite inscribable polyhedron has a perfect matching containing any two given disjoint edges.

2 Preliminaries

Except as noted, we use the graph-theoretical notation and definitions of [2]. $V(G)$ and $E(G)$ denote the set of vertices and edges of a graph G, respectively. If $S \in V(G)$, $I(S)$ denotes the set of edges incident on some vertex in S, and $N(S)$ denotes the set of all vertices adjacent to some vertex in S. If $v \in V(G)$, $I(v)$ and $N(v)$ are shorthand for $I(\{v\})$ and $N(\{v\})$, respectively. $|S|$ denotes the cardinality of a set S, and $\mathbf{deg}(v) = |N(v)|$ denotes the degree of a vertex v. A graph G is *1-tough* [6] if for all nonempty $S \subseteq V(G)$, $c(G-S) \leq |S|$. (Here $c(\cdot)$ denotes the number of connected components.) G is *1-supertough* if, for all $S \subseteq V(G)$ with $|S| \geq 2$, $c(G-S) < |S|$.

A *Hamiltonian cycle* in a graph is a spanning cycle. A graph is *Hamiltonian* if it has such a cycle. A graph is said to be *k-Hamiltonian* if removing any k vertices from it yields a Hamiltonian graph. A k-Hamiltonian graph is $(k-2)$-connected. A famous theorem of Tutte [38, 39] asserts that any 4-connected planar graph is Hamiltonian, and that there is a Hamiltonian cycle passing through any two given edges incident on a common face. A refinement due to Nelson (see [37]) says that any 4-connected planar graph is 1-Hamiltonian.

A *triangulation* is a 2-connected plane graph in which all faces except possibly the outer face are bounded by triangles. The *Delaunay tessellation*, DT(S), of a planar set of points S is the unique graph with $V(G) = S$ such that the outer face is bounded by the convex hull of S, all vertices on the boundary of a common interior face are cocircular, the vertices of an interior face are exactly the points of S lying on the circumcircle of the face, and no points of S lie in the interior of a circumcircle of any interior face. DT(S) is said to be *nondegenerate* if it is a triangulation and all convex hull vertices of S are extreme points of S, *degenerate* otherwise. If DT(S) is nondegenerate, it is called the Delaunay triangulation. Elementary properties of the Delaunay tessellation/triangulation, and the more conventional definition as the dual of the Voronoi diagram, are developed in [1, 16, 29]. We call a triangulation *Delaunay realizable* if it is combinatorially equivalent to a Delaunay triangulation.

A graph G is *polyhedral* if it can be realized as the edges and vertices of the convex hull of a noncoplanar set of points in 3-space (a *polyhedron*). A famous theorem of Steinitz (see [20]) asserts that a graph is polyhedral if and only if it is 3-connected and planar. A polyhedron is *trivalent* if all its vertices have degree 3, *simplicial* if all its faces are triangles. A polyhedron is trivalent if and only if its dual is simplicial. A polyhedron is *inscribable* if it has a (combinatorially equivalent) realization as the edges and vertices of the convex hull of a noncoplanar set of points on the surface of a sphere in 3-space. A polyhedron is *circumscribable* if it has a (combinatorially equivalent) realization as a polyhedron each of whose faces is tangent to a common sphere. Both inscribability and circumscribability are properties of combinatorial types of polyhedra (i.e., their graphs), so it is reasonable to talk about inscribable and circumscribable *graphs*. It is shown in [20] that a polyhedron is circumscribable if and only if its dual is inscribable. A *cutset* in a graph is a minimal set of edges whose removal increases the number of components. A cutset is *noncoterminous* if its edges do not all have a common endpoint. The following result is due to Rivin [30] (also see [22, 23, 32, 34]).

Characterization 2.1 *A graph is inscribable if and only if it is polyhedral and weights w can be assigned to its edges such that:*

(W1) *For each edge e, $0 < w(e) < 1/2$.*

(W2) *For each vertex v, the total weight of all edges incident on v is equal to 1.*

(W3) *For each noncoterminous cutset $C \subseteq E(G)$, the total weight of all edges in C is strictly greater than 1.*

The following lemma describes the connection between Delaunay tessellations and inscribable graphs, using a different formulation from that in [4]. The proof is an immediate consequence of basic properties of stereographic projection [8]. The operation of *stellating* a face f in a plane graph G consists of adding a vertex inside the face f and then connecting all vertices incident on f to the new vertex.

Lemma 2.2 *A plane graph G is realizable as* DT(S) *for some set S, with f as the unbounded face, if and only if the graph G' obtained from G by stellating f is inscribable.*

The following lemma, which is proved in [15], characterizes the circumstances in which adding edges to inscribable graphs preserves inscribability. Here and throughout the paper, we assume that all bipartite graphs are 2-colored red and blue.

Lemma 2.3 ([15]) *Let G be an inscribable graph. Suppose that H is obtained from G by performing any of the following transformations in such a way that H remains planar.*

(T1) *If G is nonbipartite, adding an edge to G.*

(T2) *If G is bipartite, adding a red-blue edge to G.*

(T3) *If G is bipartite, adding a red-red edge and a blue-blue edge to G.*

Then H is inscribable, and can be realized through an arbitrarily small perturbation of the vertices of G.

3 Sufficient Conditions

Theorem 3.1 *Any 1-Hamiltonian, planar graph is inscribable.*

Proof Let G be 1-Hamiltonian and planar. Since G is 3-connected, it is polyhedral. Let $v_1, \dots, v_n$ be the vertices of G. For $i \in \{1, \dots, n\}$, let Z_i be a Hamiltonian cycle through $G - \{v_i\}$. For each $e \in E(G)$, let $x_i(e) = 1$ if Z_i passes through e, 0 otherwise, and let

$$w(e) = \frac{\sum_{i=1}^{n} x_i(e)}{2(n-1)}.$$

Let H be the subgraph of G consisting of those edges e for which $w(e) > 0$. By construction, H is 1-Hamiltonian, hence polyhedral. We claim that the function w, when restricted to $E(H)$, satisfies conditions (W1)–(W3) of Characterization 2.1. Indeed, since each edge e is on at least one and at most $n-2$ of the Z_i, $0 < w(e) < (n-2)/(2(n-1))$, so (W1) is satisfied. Since each vertex of H is on exactly $n-1$ cycles, (W2) holds. Finally, every Z_i crosses each noncoterminous cutset at least twice, so the total weight across each cutset is at least $n/(n-1) > 1$. Hence H is inscribable, by Characterization 2.1.

Since H is 1-Hamiltonian, it cannot be bipartite, so adding the edges of $G - H$ to H preserves inscribability by Lemma 2.3. Hence G is inscribable. ■

Corollary 3.2 *If $k > 0$, any k-Hamiltonian planar graph is inscribable.*

Proof For $k > 0$, any $(k+1)$-Hamiltonian planar graph is necessarily k-Hamiltonian. Indeed, if G is $(k+1)$-Hamiltonian and planar, then G is $(k+3)$-connected, so removing $k-1$ vertices from G leaves a 4-connected graph. Since any 4-connected planar graph is 1-Hamiltonian, it follows that G is k-Hamiltonian. By induction, G is 1-Hamiltonian, hence inscribable by Theorem 3.1. ■

Notice that the only feasible values of k in Lemma 3.2 are 1, 2 and 3, since no planar graph can be 6-connected. Lemma 3.2 is false for $k = 0$, as there exist Hamiltonian, noninscribable polyhedra, such as the stellated tetrahedron shown in Figure 1(a).[1] Thomassen has shown that there exist 1-Hamiltonian, planar graphs that are not Hamiltonian [36].[2]

Theorem 3.3 *Any 4-connected planar graph is inscribable.*

[1] The noninscribability of the stellated tetrahedron follows immediately from Theorem 4.1, below.

[2] Thomassen's example and Theorem 3.1 jointly provide the following historical footnote. At the 1986 SCG conference in Yorktown Heights, the question was raised whether all inscribable polyhedra are Hamiltonian [27]. The question was answered in the negative in [10]: there is a 25-point counterexample. It follows from Theorem 3.1 that Thomassen's 105-point example of [36], which had been discovered 10 years before the conference, was also a counterexample.

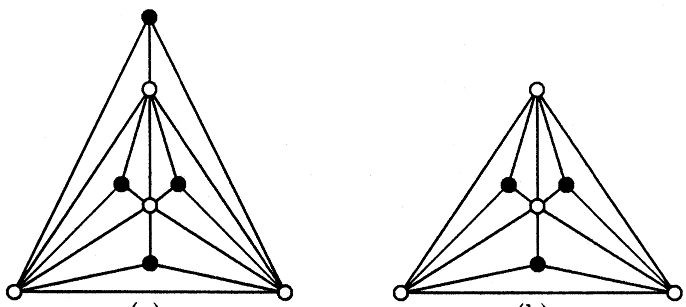

Figure 1: (a) The stellated tetrahedron is Hamiltonian, but noninscribable. (b) This graph is 1-Hamiltonian and has a Hamiltonian cycle passing through every edge, but it is not Delaunay realizable.

Proof This follows immediately from Theorem 3.1 and Nelson's theorem. ■

We note that a 4-connected graph need not be circumscribable. Examples are given on [14, page 184].

Our next goal is to show that any triangulation without chords or separating triangles is realizable as a Delaunay triangulation (Theorem 3.6). (A *chord* is an edge connecting two nonconsecutive vertices on the outer face, and a *separating triangle* is a nonfacial triangle). We first establish a more general theorem (Theorem 3.4). Before stating this theorem, we remark that it is best possible in the following sense: there exist graphs that are 1-Hamiltonian and have a Hamiltonian cycle passing through every edge but which are not realizable as Delaunay tessellations. One such example is the graph of Figure 1(b), which is not realizable as a Delaunay tessellation because the graph of Figure 1(a) is not inscribable.

Theorem 3.4 *If G is planar and 1-Hamiltonian, F is a face of G, and there is a Hamiltonian cycle of G passing through any two consecutive edges on the boundary of F, then G is realizable as a Delaunay tessellation (with outer face F).*

Proof Let G and F be as in the statement of the theorem. Let v_i, $i = 0, \dots, k-1$, be the vertices of G on the boundary of F, listed consecutively about the boundary of F. Let G' be the graph obtained by stellating face F, with v the stellating vertex. By Lemma 2.2, it suffices to prove that G' is inscribable. We construct a weighting of G' satisfying Characterization 2.1 in three steps.

Step 1: Let w be a weighting for G, satisfying conditions (W1)–(W3) of Characterization 2.1. Such a weighting exists by Theorem 3.1.

Step 2: For each $i \in \{0, \dots, k-1\}$, let Z_i be a Hamiltonian cycle of G using the edges $v_{i-1}v_i$ and v_iv_{i+1}, where the subscripts are taken modulo k. For each $i \in \{0, \dots, k-1\}$ and each $e \in E(G)$, let $y_i(e) = 1/2$ if Z_i passes through e, 0 otherwise. Each function $y_i(\cdot)$ satisfies (W2), and it also satisfies (W1) and (W3) except that the inequalities are not strict. Let

$$y(e) = \frac{w(e) + k\sum_{i=0}^{k-1} y_i(e)}{1+k^2} \tag{3.1}$$

Since y is a convex combination of w and the y_i's, y satisfies conditions (W1)–(W3). Also each edge e incident on F satisfies the inequality

$$y(e) \geq k/(k^2+1) > 1/(2k). \tag{3.2}$$

Step 3: Define a new weighting function x on $E(G')$ by:

$$x(e) = \begin{cases} y(e) & \text{if } e \in E(G) \text{ and } e \text{ is not part of the boundary of } F \\ y(e) - 1/(2k) & \text{if } e \in E(G) \text{ and } e \text{ is part of the boundary of } F \\ 1/k & \text{if } e = vv_i \text{ for some } i \end{cases}$$

It is clear that $x(\cdot)$ satisfies (W1) and (W2). Let C be any cutset in G'. If C does not contain any edges of G incident on F, then $\sum_{e\in C} x(e) = \sum_{e \in C} y(e)$. Otherwise, C contains at least one edge incident on v for every pair of edges on the boundary of F, so $\sum_{e\in C} x(e) \geq \sum_{e\in C} y(e)$. Hence $x(\cdot)$ satisfies (W3), and the proof is complete. ■

Theorem 3.5 *Any 4-connected planar graph is realizable as a Delaunay tessellation, with an arbitrary face as its outer face.*

Proof This is immediate from Theorem 3.4, Nelson's Theorem, and Tutte's theorem. ■

Theorem 3.6 *Any triangulation T without chords or non-facial triangles is realizable as a Delaunay triangulation, with the nontriangular face as the outer face.*

Proof If the outer face has valence 4 or more, then stellating the outer face yields a 4-connected graph, so the result follows from Corollary 3.3 and Lemma 2.2. If the outer face is a triangle, then T is 4-connected, so the result follows from Corollary 3.5. ■

Since no bipartite graph can be 1-Hamiltonian, the preceding theorems do not apply in the bipartite case. Define a bipartite graph to be *red-blue-Hamiltonian* if whenever a red vertex and a blue vertex are removed, the graph is Hamiltonian. Theorem 3.8, a bipartite analog of Theorem 3.1, is an immediate consequence of the following more general theorem.

Theorem 3.7 *If a planar graph G has the property that removing any pair of adjacent vertices yields a Hamiltonian graph, then G is inscribable.*

Proof Omitted; see the full paper. ■

Theorem 3.8 *If a planar bipartite graph is red-blue Hamiltonian, then it is inscribable.*

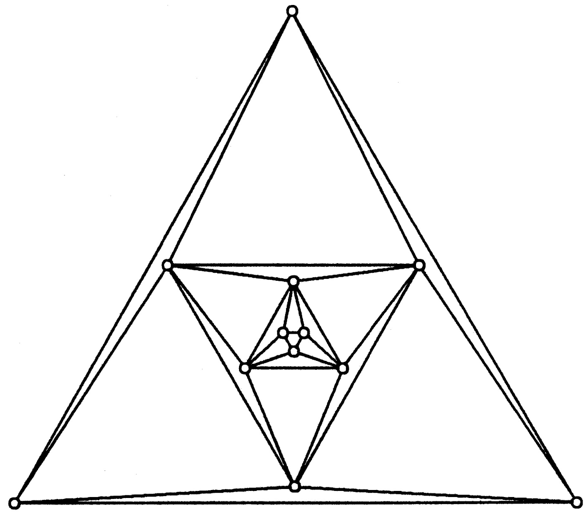

Figure 2: A "string of pearls" graph.

Proof Immediate from Theorem 3.7. ■

We conclude this section by showing that if a polyhedron is "almost regular" in a certain sense, then it is inscribable. This leads to a sufficient condition for Delaunay realizability, phrased in terms of degree constraints.

Theorem 3.9 *Every simplicial polyhedron in which every vertex has degree 4, 5, or 6 is 1-Hamiltonian and hence, by Theorem 3.1, inscribable.*

Proof [sketch] Let G be a simplicial polyhedron in which every vertex has degree 4, 5, or 6. If G is 4-connected, the result follows from Theorem 3.3. If G is not 4-connected, it is shown in the full paper that G must be a "string of pearls," a nested sequence of triangles as shown in Figure 2. Such a graph is easily seen to be 1-Hamiltonian, and hence inscribable by Theorem 3.1. ■

The "string-of-pearls" graphs just defined are not 4-connected, but their duals are bipartite and trivalent. It follows from [14, Theorem 3.1] that they are not circumscribable. So Theorem 3.9 is false if we replace "inscribable" with "circumscribable." However, we have:

Theorem 3.10 *Every simplicial polyhedron in which every vertex has degree 5 or 6 is both inscribable and circumscribable.*

Proof Inscribability is a special case of Theorem 3.9. It follows from the proof of Theorem 3.9 that if G is a simplicial polyhedron in which every vertex has degree 5 or 6, then G is 4-connected. It is observed in [14] that any trivalent polyhedron with a 4-connected dual is inscribable (proof: assign each edge a weight of 1/3). This observation implies that G has an inscribable dual, so G is circumscribable. ■

Theorem 3.11 *Suppose a triangulation T satisfies the following degree conditions:*

- *Every interior vertex has degree 4, 5, or 6;*
- *Every boundary vertex has degree 3, 4, or 5; and*
- *The outer face has degree 4, 5, or 6.*

Then T is Delaunay realizable.

Proof The conditions imply that stellating the outer face would yield a graph in which all vertices have degree between 4 and 6, inclusive, so the result follows from Lemma 2.2 and Theorem 3.10. ■

4 Necessary conditions

The following theorem is proved in [14].

Theorem 4.1 *Any nonbipartite inscribable graph is 1-supertough.*

A *perfect matching* in an n-vertex graph is a set of $\lfloor n/2 \rfloor$ disjoint edges, where $\lfloor \cdot \rfloor$ denotes the "floor" function. The following two theorems assert the existence of perfect matchings containing specified edges in inscribable polyhedra. They strengthen the results of [12], which showed that a perfect matching existed but allowed no additional constraints. Proofs, omitted here, are given in the full paper.

Theorem 4.2 *Any nonbipartite inscribable graph has a perfect matching containing any given edge.*

Theorem 4.3 *Any bipartite inscribable graph has a perfect matching containing any two given disjoint edges.*

Examples given in the full paper show that both of these theorems are best possible, and that Theorem 4.2 becomes false if we replace "nonbipartite matching" with "Delaunay triangulation."

5 Remarks

Theorems 3.1 and 4.1 provide a pair of sufficient and necessary conditions that bracket the class of inscribable graphs. Specifically, Theorem 3.1 says that if G is planar and removing any vertex from G yields a Hamiltonian graph, then G is inscribable. Theorem 4.1 says that if G is inscribable, removing any vertex from G yields a 1-tough graph. It is well known that any Hamiltonian graph is 1-tough [6]. Lemma 2.2 suggests an alternative formulation of these two theorems. Let G be any triangulation with n vertices, and let G' be the simplicial planar graph obtained by stellating the outer face of G. Consider the family $\mathcal{F}$ of $n+1$ triangulations that can be obtained by deleting a vertex of G. Theorem 3.1 says that if *every one* of the triangulations in $\mathcal{F}$ is Hamiltonian, then G' is inscribable and hence every triangulation in $\mathcal{F}$ (including G) is Delaunay realizable. Theorem 4.1 says that if G is Delaunay realizable, then every triangulation in $\mathcal{F}$ is 1-tough.

In view of the reformulation in the preceding paragraph, it is tempting to conjecture that there is some property P, between Hamiltonicity and 1-toughness, such that a nonbipartite polyhedral graph is inscribable if and only if removing any vertex produces a graph with property P. A proof of some instantiation of this statement would provide a purely graph-theoretical characterization of inscribable polyhedra (at least in the nonbipartite case), and hence provide a purely-graph theoretical characterization of Delaunay triangulations. However, it is not clear what property P might be.

The existence of a relationship between Hamiltonicity and inscribability has been previously noted. Indeed, it was observed in [9] that *any* Hamiltonian polyhedral graph is inscribable in a certain highly degenerate sense: the graph can be realized as a polyhedron, "flattened" to a disk, with all the vertices lying on a common circle in an order determined by the Hamiltonian cycle. The results of Section 3 indicate that this relationship is rather strong. Nevertheless, there are limits to the extent of the relationship. In particular, it is an NP-complete problem to determine whether an inscribable polyhedron (or a Delaunay triangulation) is Hamiltonian [13].

We close with two open problems:

1. We conjecture that any simplicial polyhedron with degree ≤ 6, with the single exception of the stellated tetrahedron, is 1-Hamiltonian and hence inscribable. This result would strengthen Theorem 3.9 by removing the lower bound on the vertex degrees. We have verified this conjecture for all simplicial polyhedra with up to 15 vertices. Ewald has shown that any simplicial polyhedron with maximum degree ≤ 6 is Hamiltonian [18].

2. The methods of Section 3 are, in principle, constructive. In particular, a weighting of a 4-connected polyhedron satisfying conditions (W1)–(W3) can be found in quadratic time by repeatedly using algorithms from [5]. Once such a weighting is known, an inscription can be found in polynomial time [31, 33]. Nevertheless, it would be useful to have faster methods for directly constructing inscriptions and Delaunay realizations of these polyhedra.

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