Tolerance-Free NeST Representations*

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Abstract. We establish a tolerance-free characterization of NeST graphs, and introduce two new subclasses. We show that fixed diameter (unit) NeST graphs are exactly proper NeST graphs, and that fixed distance NeST graphs are exactly threshold tolerance graphs. The latter answers a question of Monma, Reed and Trotter, and implies polynomial recognition of fixed distance NeST graphs. We also answer a question of Bibelnieks and Dearing by showing that not all NeST graphs are weakly triangulated.

1. Introduction. In 1957 Häjos introduced *interval graphs*, namely intersection graphs of intervals on a line [Hj]. In 1982 Golumbic and Monma introduced *interval tolerance graphs*, or simply *tolerance graphs* [GM], described as follows:

> *Two vertices are adjacent if and only if the size of the overlap of their intervals (on a line) exceeds at least one of their tolerances.*

In 1991 Bibelnieks and Dearing generalized interval tolerance graphs to *neighborhood subtree tolerance* (NeST) *graphs* [BD]. NeST graphs generalize interval tolerance graphs by generalizing intervals on a line to neighborhood subtrees of a plane-embedded tree, where a neighborhood subtree is specified by its center and radius. Thus, NeST graphs are described as follows:

> *Two vertices are adjacent if and only if the size of the overlap of their neighborhood subtrees (in an embedded tree) exceeds at least one of their tolerances.*

Why generalize to NeST graphs? Generalizing interval graphs by replacing the line model with a tree has been studied (yielding intersection graphs of subtrees of a tree, or *subtree graphs*); adding tolerances to this model is an obvious further step. However, as Bibelnieks and Dearing point out, when this step is taken, all graphs are realized; thus some restriction must be put on the subtrees permitted. The restriction they choose is that of the neighborhood subtree, studied earlier by Tamir in relation to certain planar location problems [T].

Inclusion relations for the classes discussed so far are shown in Figure 1. (*Neighborhood graphs* are intersection graphs of neighborhood subtrees in an embedded tree.) In this figure, the top three classes are intersection graph classes; the bottom three are obtained by introducing tolerances.

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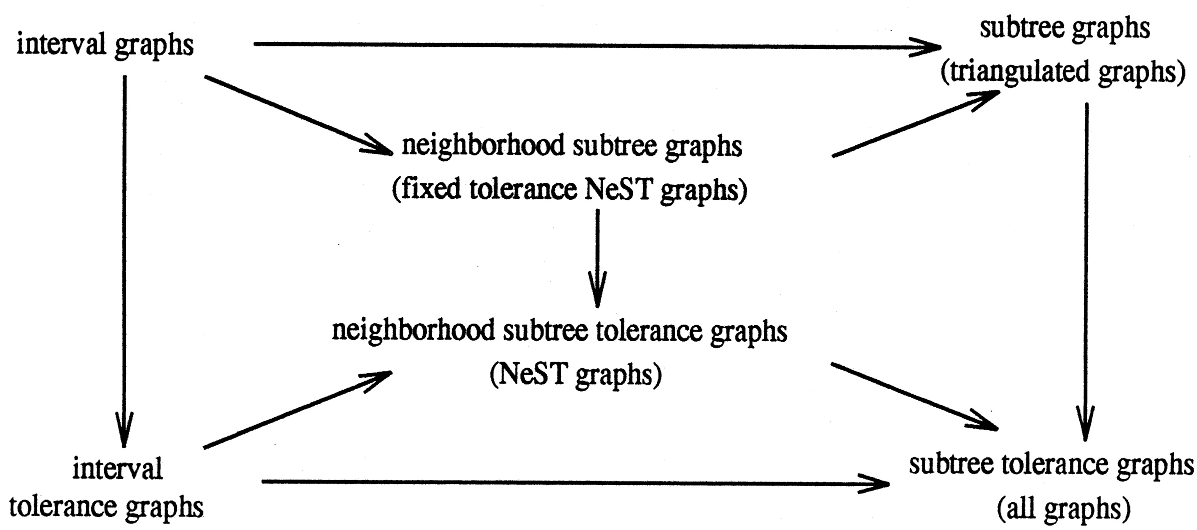

FIGURE 1. *Generalizing interval graphs and interval tolerance graphs (arrows show set inclusion).*

Our main result is a characterization of NeST graphs in which tolerances do not explicitly appear, but are instead implicitly determined by other parameters. This characterization has several consequences, which we mention later.

2. Defining NeST Graphs. To define NeST graphs precisely, we must define the notions of embedding tree, neighborhood subtree, and neighborhood subtree size.

Let $\mathcal{T}$ be a tree and let T be an embedding of $\mathcal{T}$ in the plane. Let $P(x,y)$ be the unique path in T between points x and y, and $d(x,y)$ the length of $P(x,y)$. Denote by $T(c,r)$ the *neighborhood subtree* of T with center $c \in T$ and radius $r \geq 0$, defined as the set of points $\{x \in T : d(x,c) \leq r\}$.

Now we define neighborhood subtree size. Let S be any connected subset of points of an embedded tree T. Define the function $|\cdot|$ so that

$$|S| = \begin{cases} max\{d(p_1,p_2) : p_1,p_2 \in S\} & \text{if } S \neq \phi \\ 0 & \text{if } S = \phi. \end{cases}$$

For any neighborhood subtree $T(c,r)$, call $|T(c,r)|$ its *diameter.* Thus $|T(c,r)|$ is the length of a longest path in $T(c,r)$.

DEFINITION 2.1. *A graph G is a* neighborhood subtree tolerance (NeST) graph *if there exists an embedded tree T, a set $S = \{T(c_v, r_v) : v \in V(G)\}$ of neighborhood subtrees of T and a set $\mathcal{T} = \{\tau_v : v \in V(G)\}$ of positive numbers called*

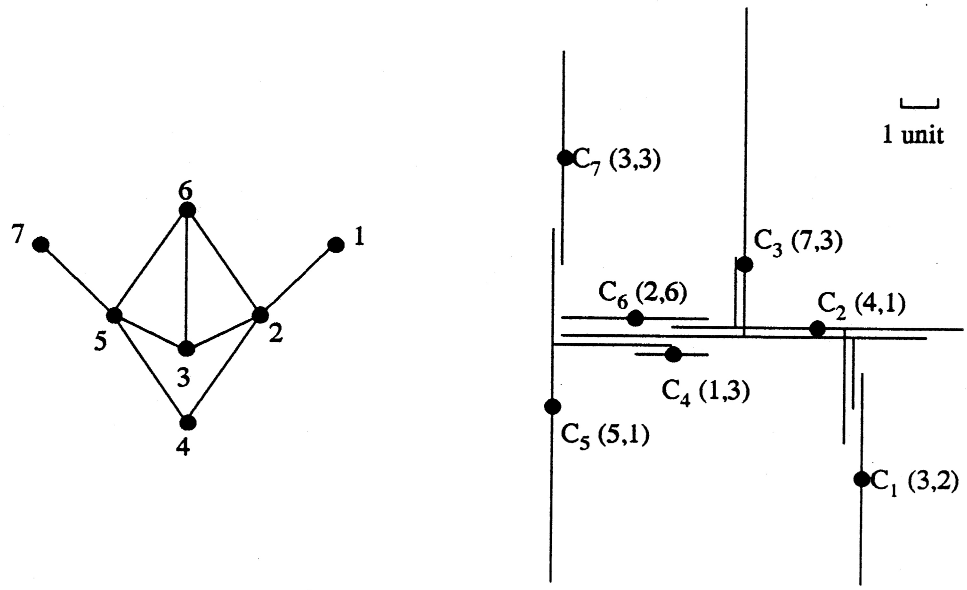

FIGURE 2. *A graph and its NeST representation. Pairs denote radius and tolerance values.*

tolerances *such that the following edge condition holds:*

$$xy \in E(G) \Leftrightarrow |T(c_x, r_x) \cap T(c_y, r_y)| \geq min\{\tau_x, \tau_y\}.$$

The triple $(T, S, \mathcal{T})$ *is called a* neighborhood subtree tolerance (NeST) representation *of* G*, and* G *is called the graph* associated *with the NeST representation* $(T, S, \mathcal{T})$.

We use T_x as shorthand for the neighborhood subtree $T(c_x, r_x)$, and T_{xy} for $T_x \cap T_y$. In referring to a NeST representation $(T, S, \mathcal{T})$, it is assumed that T_x, c_x, r_x and τ_x denote the representation's neighborhood subtree, center, radius and tolerance respectively, for each vertex x of the associated graph.

3. The Main Result. We obtain a characterization of NeST graphs in which tolerances do not appear explicitly. Given a graph G with NeST representation $(T, S, \mathcal{T})$, define $B(x)$ for every vertex $x \in V(G)$ as

$$B(x) = \{z \in M(x) : |T_{xz}| \geq |T_{xy}|, \text{ for all } y \in M(x)\},$$

where $M(x)$ denotes the vertices of $G - x$ that are not adjacent to x. Thus $B(x)$ is the set of non-neighbors of x that maximize the size of their neighborhood subtree intersections with the neighborhood subtree of x.

DEFINITION 3.1. *The pair* (T, S)*, where* T *is an embedded tree and* S *is a set of neighborhood subtrees in* T*, is a* tolerance-free NeST representation *of the graph* G

if the following edge condition holds:

$$xy \in E(G) \Leftrightarrow |T_{xy}| > min\{|T_{x\dot{x}}|, |T_{y\dot{y}}|\},$$

where for a vertex z, $\dot{z}$ is any vertex in $B(z)$, with $|T_{z\dot{z}}| = 0$ whenever $B(z)$ is empty.

Stating that G is a NeST graph is equivalent to stating that there is a NeST representation $(T, S, \mathcal{T})$ such that G and $(T, S, \mathcal{T})$ satisfy the following relation:

$$xy \in E(G) \quad \Leftrightarrow \quad |T_{xy}| \geq min\{\tau_x, \tau_y\}.$$

Our main result is the following theorem, which reveals the role of tolerances in a NeST representation as placeholders. One direction of the (omitted) proof is an algorithm for constructing tolerances from a tolerance-free representation.

THEOREM 3.2. *$(T, S, \mathcal{T})$ is a NeST representation of G if and only if (T, S) is a tolerance-free NeST representation of G.* □

We point out that while a NeST representation is satisfied by only one graph, a tolerance-free NeST representation may be satisfied by several.

4. Consequences. The above theorem is our main result not in the sense of being most significant, but rather of being primary. Without stating any proofs, we summarize several consequent results.

We consider four subclasses of NeST graphs whose definitions follow from natural restrictions of the definition of a NeST graph, namely

- *fixed diameter NeST graphs*, in which all neighborhood subtrees have the same diameter,
- *proper NeST graphs*, in which no neighborhood subtree is properly contained in another,
- *fixed distance NeST graphs*, in which neighborhood subtree centers are equidistant and
- *fixed tolerance NeST graphs*, in which all neighborhood subtrees have the same tolerance,

The first and third of these subclasses are new; the other two have been previously studied [BD].

Fixed diameter graphs are to NeST graphs as unit interval graphs are to interval graphs, and as unit interval tolerance graphs are to interval tolerance graphs. For interval and interval tolerance graphs it has been asked whether the respective proper and unit subclasses are equivalent. We ask the analogous question of NeST graphs and answer in the affirmative. That is, proper NeST graphs are equivalent to unit NeST graphs. Consequently, this yields a characterization of proper NeST graphs that is both radius-free and tolerance-free.

We obtain a characterization of fixed distance NeST graphs that depends only on the radii of the neighborhood subtrees, namely, the characterization is both tolerance-free and center-free. Via this characterization we show that the class of fixed distance NeST graphs is exactly the class of threshold tolerance graphs. This responds to a question asked by Monma, Reed and Trotter. The equivalence of the two classes implies polynomial recognition of fixed distance NeST graphs by the threshold tolerance graph recognition algorithm presented in [MRT].

Fixed tolerance NeST graphs are exactly the intersection graphs of neighborhood subtrees of a tree [T]. We obtain a characterization of fixed tolerance NeST graphs that depends only on the location of the neighborhood subtree centers within the embedded tree.

Finally, in response to a question of Bibelnieks and Dearing, we prove that there are graphs that are weakly triangulated but not NeST.

To conclude, here are some open problems:

- Are all NeST graphs proper NeST graphs?
- Are all triangulated graphs NeST graphs?
- For each of fixed tolerance graphs, proper NeST graphs, and NeST graphs, what is the complexity of recognizing this class of graphs?

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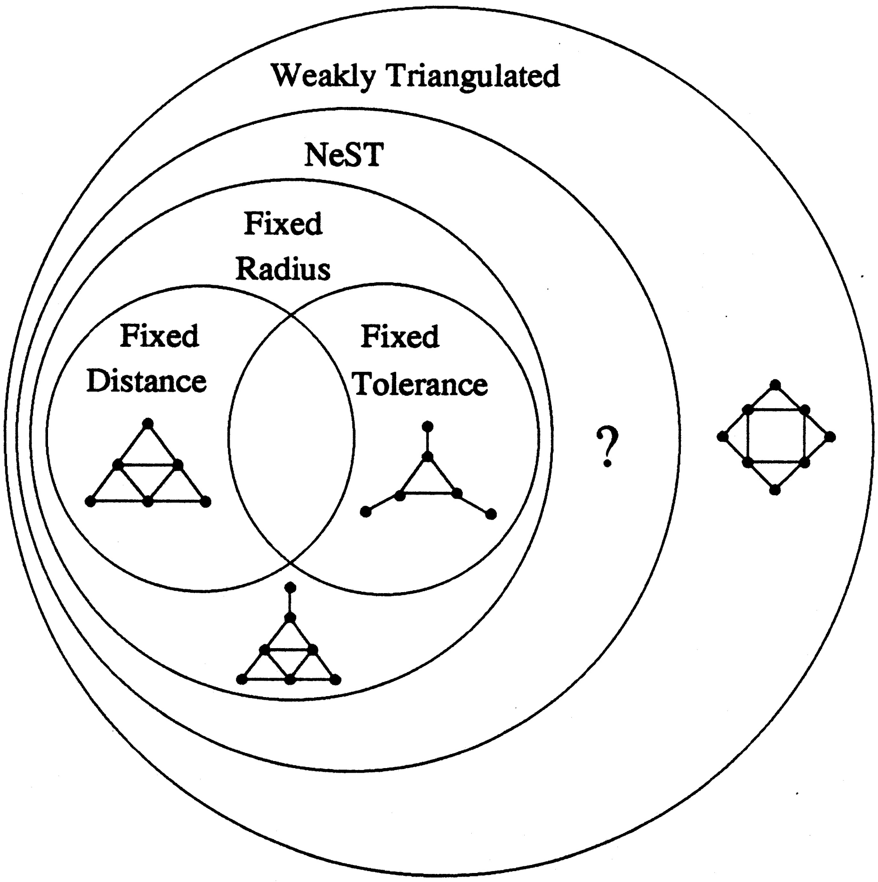

FIGURE 3. *Subclasses and superclasses of NeST graphs*