Tolerance of Geometric Structures *

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Abstract

In this paper we define the concept of *tolerance* of a geometric or combinatorial structure associated to a set of points as a measure of how much the set of points can be perturbed in order that the structure remains essentially (topologically or combinatorially) the same and we give some examples of the variety of problems suggested by this definition.

1 Introduction

Let $S = \{p_1, \ldots, p_n\}$ be a set of points in the plane and let us consider a *discrete* geometric structure associated to this set of points. If S is in *general position* (the meaning of general position depends on the structure under consideration), we can move the points arbitrarily inside some neighbourhood (perhaps very small) and be certain that the structure remains *topologically* or *combinatorially* the same.

The *tolerance* of the structure is defined as the supreme of $\varepsilon \geq 0$ such that if each point p_i is moved arbitrarily but not more than ε then the structure does not change. In this paper, more than giving details of how to compute the tolerance for specific structures, we focus on describing the concept, its variations and its applications.

Consider for instance the Delaunay triangulation associated to the set S, $DT(S)$. If the points of S are in general position (no four cocircular points with empty circunscribing circle, no three colinear points in the convex hull) and if ε is small enough, we know that we can move the points of S arbitrarily but not more than ε and be sure that the Delaunay triangulation does not change. We define the tolerance of $DT(S)$ as the supreme of such ε. We are looking for the supreme of ε such that each point p_i can be arbitrarily moved in the disk centered at p_i and with radius ε without producing any change in the Delaunay triangulation (see Figure 1).

More formally, let $S = \{p_1, \ldots, p_n\}$ and $S' = \{p'_1, \ldots, p'_n\}$ be two sets of n labeled points and define

$$\delta(S, S') = \max_{i=1,\ldots,n} d(p_i, p'_i). \tag{1}$$

It is easy to prove that δ is a distance between labeled sets of n points. We shall say that S' is a $\varepsilon - perturbation$ of S if $\delta(S, S') = \varepsilon$.

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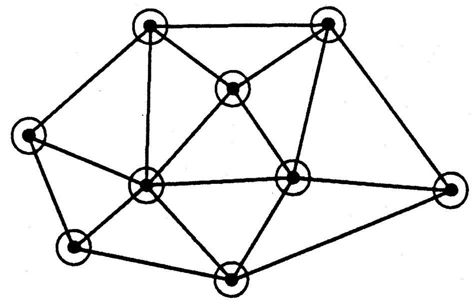

Figure 1: $DT(S)$ does not change if points move inside the disks.

We define the tolerance of $DT(S)$ as follows:

$$tol(DT(S)) = sup\ \{\varepsilon \geq 0 \mid DT(S) \sim DT(S')\ \forall\, S' \text{ such that } \delta(S, S') \leq \varepsilon\}, \qquad (2)$$

where $DT(S) \sim DT(S')$ means that p_i and p_j are neighbours in $DT(S)$ if and only if p'_i and p'_j are neighbours in $DT(S')$.

The same question make sense not only with sets of points, but also with more complicated objects. The only thing we need is a measure for perturbations of the set. For instance, let S be a set of segments and let $\mathcal{A}(S)$ be the arrangement generated by S. We can define a measure for perturbations of S by labeling the endpoints of the segments and then taking the maximum among the perturbations of the endpoints. Now we can define the tolerance of $\mathcal{A}(S)$ as the supreme of ε such that the combinatorial structure of the arrangement is the same for all ε-perturbations of S (see Figure 2).

If $\mathcal{P}$ is a simple polygon, we can define a measure for perturbations of $\mathcal{P}$ by labeling the vertices of the polygon and then taking the maximum among the perturbations of the vertices. Actually, the tolerance was first introduced in [1] where the tolerance of the simplicity of a (simple) polygon was defined and computed using Voronoi diagrams.

Another class of problems appears when the structure is not unique. Consider for instance a

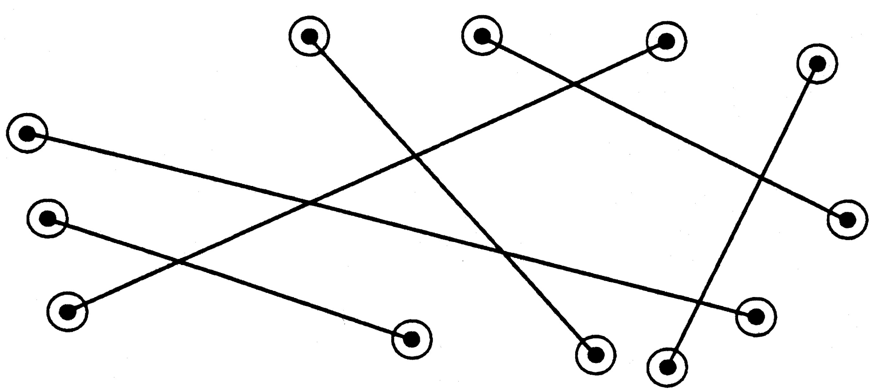

Figure 2: $\mathcal{A}(S)$ does not change if points move inside the disks.

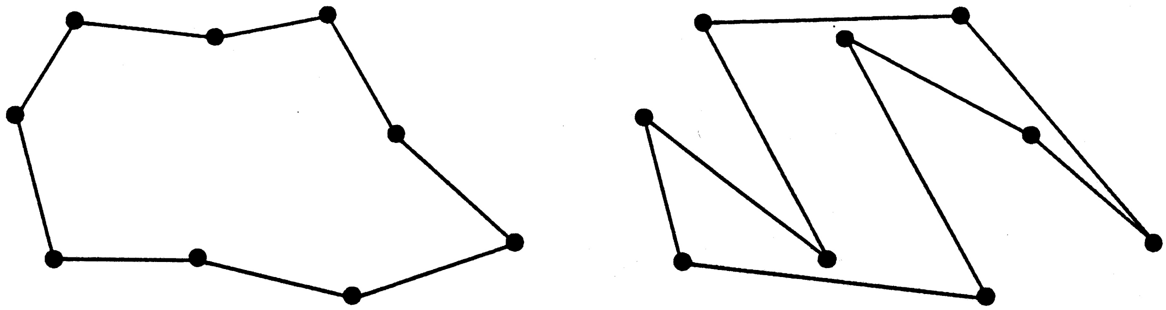

Figure 3: Two different polygonizations with very different tolerance of their simplicity.

set S and a line l that bisects S. We can define the tolerance of the property "l is a bisector of S" as usual. We can interpret the tolerance as a *quality coefficient* for the bisector and then try to compute the "best" bisector for S. This is an example of *optimization* problem, a variation that can lead to hard problems.

For instance, let S be a set of points. Given a polygonization of S, we can compute the tolerance of the simplicity of the resulting polygon and interpret it as a quality coefficient for the polygonization (see Figure 3). Then the following problem can be proposed: compute the "best" polygonization of a set of points.

Another class of problems arises when we ask about the set that maximizes the tolerance of a certain structure or property (of course scaling the problem). If $\mathcal{P}$ is a convex polygon, we can define the tolerance of this property as usual. If we divide the tolerance by a measure of the size of the polygon, namely, the radius of the minimum enclosing circle, then we obtain something that can be interpreted as a measure of convexity. The question is now: what is the "most convex" n-gon that can be enclosed in the unit disk? In a similar way, we can interpret the tolerance of the simplicity as a measure of simplicity and propose the problem: what is the "simplest" n-gon that can be enclosed in the unit disk?

2 Applications and prior related work

The tolerance is naturally related to the accuracy of the input of data since, if the tolerance is big, errors comparatively small in the input will be irrelevant. On the contrary, if the situation is like in Figure 4 (where the tolerance is small), even tiny errors in data can produce different results. However, the tolerance should not be confused with the concept of *algorithmic robustness* which studies how small roundoff errors can accumulate during different steps of an algorithm and produce a false final result. This is the approach in [6] where a concept similar to tolerance is defined, but from the point of view of algorithmic robustness. The same can be said about the papers [7] and [10] where the authors propose some algorithms to compute an approximate convex hull taking into account roundoff errors. The main difference is that the tolerance measures the possible changes of a combinatorial structure *exactly* associated with a set of points.

More related to our work are the papers [11] where the authors define the *sensitivity* of a set of points that we can refer now as the tolerance of the euclidian minimum spanning tree of the set and [13] where the author defines measures for perturbations of arrangements of lines and

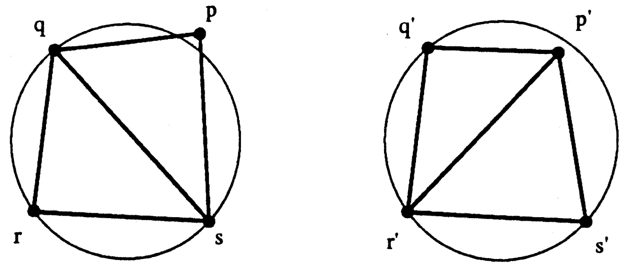

Figure 4: A small perturbation of S can change $DT(S)$.

circles and then computes the tolerance for these arrangements.

The concept of tolerance, because so natural, has already appeared implicitely in different settings. For example in [4] the authors consider polygons subjected to an assumption that we can describe now as a fixed lower bound for the tolerance of the simplicity of the polygon.

In graph layout, it is sometimes interesting to redraw a graph with slight changes to make the picture clearer while preserving the *mental map* of the diagram; a possible way to do it is to preserve some geometric graph, as the Delaunay triangulation, of the points that correspond to the nodes [5], [8], [9]; so the tolerance would give here bounds for safe perturbations of the nodes.

The tolerance can also be interpreted as a measure of how far is a configuration of points from being degenerate with respect to some geometric or combinatorial structure since degenerate configurations have tolerance equal to zero (an arbitrarily small movement of the points can change the structure).

Another framework closely related with tolerance is dynamic maintenance of geometric or combinatorial structures of moving points. Some work has been done when we have a set of moving points which trajectories can be parametrized by algebraic functions of time [3]. If we allow arbitrary motion of the points unknown in advance, then the problem is hopeless. Nevertheless, under the reasonable assuption that the velocity of the points is bounded, something can be said about the *first possible change* in the structure: if we know that the velocity of the points is bounded by k, the configuration is not degenerate for $t = 0$ and the tolerance for this configuration is ε, then no change can occur before $t = \frac{\varepsilon}{k}$.

We have mentioned several directions of research that are suggested by the concept of tolerance and a variety of problems that can be proposed. A number of them have been solved as part of one of the author's dissertation [12]. In the rest of this paper we are going to give a sketch of some examples. Details are omitted due to space limitations. An extended version for the tolerance of the Delaunay triangulation can be found in [2].

3 A small sample of results

Let $CH(S)$ be the convex hull of the set S . We define the tolerance of $CH(S)$ as

$$tol(CH(S)) = sup\ \{\varepsilon \geq 0 \mid CH(S) \sim CH(S')\ \forall S' \text{ such that } \delta(S, S') \leq \varepsilon\}, \tag{3}$$

where $CH(S) \sim CH(S')$ means that $CH(S)$ is described by the list of vertices $\{p_{i_1}, \dots, p_{i_k}\}$ and $CH(S')$ is described by the list of vertices $\{p'_{i_1}, \dots, p'_{i_k}\}$.

In order that a perturbation of S maintains $CH(S)$, we must be sure that there are no new points in $CH(S)$ and that all the points that were in $CH(S)$ remain there. The last condition can be easly verified by checking every three consecutive vertices in the convex hull. For the former one, it is not hard to see that the internal point closest to the convex hull has to be a neighbor in the Delaunay triangulation of an extremal point. Because of that, after computing the Delaunay triangulation we can compute the tolerance of the convex hull in linear time. Therefore we have a $O(n \log n)$ algorithm and moreover we can prove that this is asintotically optimal.

We recall from (2) the definition for the tolerance of the Delaunay triangulation. In order to compute $tol(DT(S))$, the first observation is that $tol(DT(S)) \leq tol(CH(S))$. In order to prevent changes in the inner edges, we have to analyze each pair of adjacent triangular faces in $DT(S)$ since as is well known, changes in the Delaunay triangulation are always diagonal flips. Once we solve the problem -given four points, compute the smallest perturbation that makes the four points cocircular-, we have a $O(n \log n)$ algorithm for computing $tol(DT(S))$ and again this is asintotically optimal.

Consider now the problem of computing the tolerance of an arrangement of n segments. If we make the assumption that all the faces of the arrangement are unbounded, then we can have at most a linear number of intersections and we have a $O(n \log n)$ algorithm for computing the tolerance of these arrangement. A straightforward generalization leads to a $O(n \log n)$ algorithm for computing the tolerance of the simplicity of a polygon. For the general case, we can prove that we only need to check bounded faces with three edges and if we have k intersections the algorithm runs in $O((n+k)\log n)$ time.

4 Concluding remarks

It is worth noticing that if $DT(S)$ is given, then $tol(DT(S))$ can be computed in linear time. As a particular case, if the points are moved away from their position less than the tolerance, the new tolerance can be computed in linear time.

Half the distance between the closest pair of points of S is obviously a lower bound for $tol(DT(S))$. So the presence of a small cluster of points highly concentrated will result in a very small value for $tol(DT(S))$. On the other hand, this cluster could not be truly relevant to the situation that $DT(S)$ is helping to describe. It is then reasonable to introduce a concept of *local tolerance* relative not to the full structure but to some subset ob edges or faces. These ideas, as well as the variants mentioned in the introduction are developped in [12] for $DT(S)$ and many other structures.

The optimization of tolerance when the structure is not unique or when we are allowed to choose the set of points seems to origine the hardest problems. We conclude with two open problems exemplifying these two situations.

The tolerance of the simplicity of a (simple) polygon is defined as the supreme of $\varepsilon \geq 0$ such that if the vertices move less than ε then the polygon remains simple. The corresponding optimization problem is the following: how should a set of points be polygonized in order that the resulting polygon has maximum tolerance?

Let us consider now a variable but normalized S. How should we choose n points inside the unit disk in such a way that the Delaunay triangulation they define has maximum tolerance?

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