CONSTRUCTING OPTICALLY IN CONSTANT TIME THE K–LEVEL OF AN ARRANGEMENT OF LINES IN THE PLANE AND OTHER RELATED PROBLEMS

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Abstract

The problem of constructing optically the k–level of an arrangement of lines in the plane optically in constant time is considered and solved. Other related problems such as constructing a weak separator of two point sets in the plane, constructing a centerpoint of a point set in the plane, etc. are also addressed and constant time optical algorithms are proposed to solve them.

1 Introduction

In spite of the enormous achievements in the development of electronic computers over the past three decades, there has always been an interest in optical computers as a potential alternative computing technology for two main reasons:

• the practical speed of an electronic signal on a silicon chip is roughly one-fifth the speed of light (in wires it slows down even more)[J];

• optics provides free space interconnection because light rays, unlike wires, can penetrate each other.

These advantages of optical computers both in the physical speed of computation and massive parallelism of interconnections as compared with electronic computers have been the main driving force in efforts to create such a computer.

However, on the one hand, there exists another measure of speed of computation – the time-complexity of algorithms – and the advantage of optical computers from the standpoint of this measure is much less obvious or not obvious at all and, hence, requires thorough investigation. On the other hand, massive parallelism of optics appears to be too rich for free space interconnection to be its only manifestation. Therefore, it is prudent to search for other manifestations of massive parallelism of optics which whould give rise to reduction in the time-complexity of algorithms.

Attempts at doing so have already yielded their first results and have given rise to the development of the first constant time optical algorithms for various problems of computational geometry [K91, K92, KS92, KS93].

This paper continues the investigation into optical computational geometry and proposes constant time optical algorithms for the following problems:

• Constructing the K-level of an arrangement of lines in the plane;

• Constructing a weak separator of two point sets in the plane;

• Constructing a bisector of a finite set of points and ham-sandwich cuts in two dimensions;

• Constructing the centerpoint of a point set in the plane;

• Constructing a separator line for a set of segments in the plane.

As in the previous papers on optical computational geometry, we assume here that the following operations can be performed optically in constant time:

(i) Minkowski sum and difference of two plane figures [K91];

(ii) Union, intersection and subtraction of plane figures [K91];

(iii) The standard duality transform that maps a point (a,b) to a line $y=-a\cdot x+b$ and a line $y=c\cdot x+d$ to a point (c,d) [K92, KS92]. It is well known that this duality preserves incidence between points and lines, and maps a point lying above (resp. below) a line l to a line lying above (resp. below) a dual point of l.

(iv) Hough duality transform that maps a point (p,θ) to a straight line $x\cdot\cos\theta+y\cdot\sin\theta=p$, and a straight line segment

$$x\cdot\cos\theta+y\cdot\sin\theta=p,\quad \theta\in[\theta_1,\theta_2]$$

to a point (p,θ) whose intensity is proportional to the length of a segment [ED];

(v) Semi-conformal changes of coordinate [B]:

$$\begin{cases} u=u(x,y);\\ v=v(x,y),\end{cases}$$

satisfying the condition

$$\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}.$$

(vi) Intensity inversion of an image: $g(x,y)=I_{max}-f(x,y)$, where $f(x,y)$ and $g(x,y)$ are intensity distributions of an image before and after inversion respectively, and $I_{max}\geq f(x,y)$ for all x,y [OHG];

(vii) Thresholding of an image at a given level of intensity (as well as extracting regions of its maximum/minimum intensity) [GCHHZY].

Remark: In what follows we use two types of thresholding: $Threshold_{\leq a}(S)$ which denotes that portion of an image S whose intensity is not greater than a, and $Threshold_a(S)$ which denotes that slice of S whose intensity is a exactly. Obviously,

$$Threshold_a(S)=Threshold_{\leq a}[I_{max}-Threshold_{\leq I_{max}-a}(I_{max}-Threshold_{\leq a}(S))].$$

2 Constructing the k-level of an Arrangement of Lines in the Plane

Let $A(L)$ be an arrangement of a set L of N lines in the plane. Recall, that the lower/upper level of the point p in the plane is defined as the number of lines in $A(L)$ which lie below/above p, not including the line(s) passing through p itself. Analogously, the lower k-level of $A(L)$, for $1\leq k\leq N$, is defined as the set of points p with

$$a(p)\leq k-1 \text{ and } b(p)\leq N-k,$$

where $a(p)$, $b(p)$ are the number of lines l in L such that p is in the (open) halfplane l^+, and l^-, respectively.

The upper k-level of an arrangement of lines is defined analogously, as the set of points p with

$$b(p)\leq k-1 \text{ and } a(p)\leq N-k.$$

Our approach to constructing the k-level of an arrangement of lines optically is based on the following definition and lemma:

Definition 2.1 *Let $\bigcup_{i=1}^{N}A_i$ be the union of N planar sets. We define the intensity of the union at point p as the number of sets A_i containing p:*

$$intensity(p,\bigcup_{i=1}^{N}A_i)=\sharp\{i|p\in A_i\}.$$

(We note that, when computing union of sets optically, by adding up their intensity distributions which are equal to their characteristic functions, the intensity as just defined corresponds to actual optical intensity.)

Lemma 2.2 *The upper k-level of an arrangement of a set $L=\{l_i\}_{i=1}^{N}$ of N lines coincides with the upper envelope of that portion of $\bigcup_{i=1}^{N}(l_i^+)^c$ which has intensity k, except at the intersection points of the lines. Analogously, the lower k-level of the arrangement concides with the lower envelope of that portion of $\bigcup_{i=1}^{N}(l_i^-)^c$ which has intensity k, except at the intersection points of the lines.*

Proof. Indeed, let $intensity(p,\bigcup_{i=1}^{N}(l_i^+)^c)=k$. Thus,

$$intensity(p,\bigcup_{i=1}^{N}(l_i^+)^c)=\sharp\{i|p\in(l_i^+)^c\}=$$

$$= \sharp\{i|p \in l_i^- \vee p \in l_i\} = \sharp\{i|p \in l_i^-\} + \sharp\{i|p \in l_i\} =$$

$$b(p) + \sharp\{i|p \in l_i\} = k.$$

Hence

$$b(p) = k - \sharp\{i|p \in l_i\}.$$

However, p lies on at least one line l_i because p belongs to the above upper envelope. Hence $\sharp\{i|p \in l_i\} \geq 1$ and $b(p) \leq k-1$.

On the other hand,

$$a(p) = \sharp\{i|p \in l_i^+\} = N - \sharp\{i|p \in (l_i^+)^c\} =$$

$$N - intensity(p, \bigcup_{i=1}^{N}(l_i^+)^c) = N - k.$$

Since we proved that both $b(p) \leq k-1$ and $a(p) = N-k$, p belongs to the upper k-level of $A(L)$.

Let us now prove that if, conversely, p belongs to the upper k-level of $A(L)$ and is not an intersection point of the lines, then $intensity(p, \bigcup_{i=1}^{N}(l_i^+)^c) = k$.

Indeed, since p is not an intersection point, then $a(p) + b(p) = N-1$ and, hence, $b(p) = k-1$, $a(p) = N-k$. Thus,

$$intensity(p, \bigcup_{i=1}^{N}(l_i^+)^c) =$$

$$b(p) + \sharp\{i|p \in l_i\} = b(p) + 1 = k.$$

Similarly, we can see that no point vertically above p can have intensity k, thus p lies on the desired upper envelope.

The proof for the lower k-level is analogous.

□

Based on this lemma, we can propose the following algorithm for constructing the upper k-level of an arrangements of lines $L = \{l_i\}_{i=1}^{N}$ in the plane:

Step 1. Compute

$$L + ray\_down = \bigcup_{i=1}^{N}(l_i^+)^c,$$

where ray_down is the binary image storing the negative half of the y-axis.

Step 2. Compute

$$Belt_k = Threshold_k(\bigcup_{i=1}^{N}(l_i^+)^c)$$

and construct, thereby, all points in the plane which have the upper k-level except at intersection points of the lines $L = \{l_i\}_{i=1}^{N}$.

Step 3. Construct that portion of $Belt_k$ which is visible from the point at $y = \infty$. As a result, we obtain the desired upper k-level except at those points which are intersection points of the lines constituting the arrangement.

To augment the image obtained by the finite number of missing points and complete thereby the construction of the upper k-level of an arrangement, we can proceed as follows:

Step 4. Compute a lower bound 2ϵ on the minimum distance between any pair of the intersection points of the lines. (We can do this with the help of the algorithm described in [KS93].)

Step 5. Construct all possible segments S_ϵ of length ϵ emanating from the origin, which are parallel to the segments of the upper k-level of the arrangement, as just constructed (the details of performing this are described in [KS93]).

Step 6. Construct the Minkowski sum of S_ϵ with the portion of the upper k-level already constructed, and extract that portion of the sum which has infinite intensity.

It is easily seen that the image obtained contains those and only those intersection points of the lines of L which belong to the upper k-level. Hence, we can proceed as follows:

Step 7. Construct the intersection of the image obtained at step 6 with the entire set of intersection points of the lines of L. As a result, we obtain the missing points of the desired upper k-level.

Step 8. Complete the construction by taking the union of those portions of it obtained at steps 3 and 7 respectively.

Hence, we obtain

Theorem 2.3 *The k-level (both lower and upper) of an arrangement of lines in the plane, as well as a region between the k-level and (k + 1)-level, can be constructed optically in constant time using a single optical processor which works in the monochromatic mode.*

3 Weak separation of two point sets in the plane

In this section we propose an optical solution for the following problem posed in [ERK].

Let $B = \{b_i\}_{i=1}^{N}$ and $R = \{r_i\}_{i=1}^{M}$ be sets of blue and red points in the plane respectively. A line l is called a strong separator of B and R if all points of B lie in one closed half-plane defined by l, and all points of R lie in the other closed half-plane defined by l. Since, a strong separator need not exist, the notion of a weak separator has been introduced in [ERK]. A line l is a weak separator if the minimum of $|B \setminus l^+| + |R \setminus l^-|$ and $|B \setminus l^-| + |R \setminus l^+|$ is minimized.

To describe the idea behind constructing a weak separator optically, we should note the following.

Let $D(B)$ be dual lines of the point set B and let $D(l)$ be the dual point of a line l. Obviously, l^+ contains k_b points of B iff $D(l)$ belongs to the locus of points lying either on the upper k_b-level of the arrangement $A(D(B))$ or between the upper k_b- and $(k_b + 1)-$levels of the arrangement.

Obviously, this locus has intensity k_b in the image $D(B) + ray\_down$.

Analogously, half-plane l^- contains k_r points of R iff $D(l)$ has intensity k_r in the image $D(R) + ray\_up$.

Since, intensities of two images are added when they are superimposed (united), $|B \setminus l^+| + |R \setminus l^-| = k$ iff $D(l)$ has intensity k in the image

$$[D(B) + ray\_down] \cup [D(R) + ray\_up].$$

Hence, the locus of points of minimum intensity in this image is the locus of dual points of all possible weak separators between B and R.

Thus, we can propose the following algorithm for constructing a weak separator optically:

Step 1. Dualize the point sets B and R to lines $D(B)$ and $D(R)$ respectively.

Step 2. Construct images $D(B) + ray\_down$ and $D(R) + ray\_up$ on separate SLMs.

Step 3. Unite the images obtained and, thus, construct the image $[D(B) + ray\_down] \cup [D(R) + ray\_up]$.

Step 4. Extract that portion of this image which has minimum intensity. As a result we obtain the dual points of all possible weak separators.

To construct the weak separator itself, we choose a point from an image obtained at step 4 and dualize it to the desired separator. This procedure of choosing optically a point from a given point set deserves the special consideration given in the following subsection.

3.1 How one chooses optically a point from a point set in constant time

In the problems we consider in this paper there are no restrictions on the point chosen. We are allowed to choose any point from a given set. Therefore, in what follows we will always choose the lowest point of all the leftmost points of a given point set P merely because this point can readily be extracted optically in constant time as follows:

Step 1. If P is empty, which can be determined by comparing intensities of light before and after passing through a SLM representing P, then quit; else continue.

Step 2. Compute all leftmost points of P as follows:

$$leftmost\_points = Threshold_{\leq 1}(P + ray),$$

because only leftmost points in the image $P + ray$ have unit intensity and there are no points that have intensity of less than 1.

Step 3. Compute the desired point as follows:

$$lowest\_leftmost\_point =$$

$$Threshold_{\leq 1}(leftmost\_points + ray\_up).$$

Thus, we obtain

Lemma 3.1 *A point from a non-empty point set can be chosen optically in constant time by a single optical processor which works in monochromatic mode.*

Taking this result and the previous algorithm into accpunt, we obtain

Theorem 3.2 *A weak separator of two point sets in the plane can be constructed optically in constant time by a single optical processor which works in monochromatic mode.*

Remark: As pointed out in [ERK], the problem of constructing a weak separator is equivalent to finding a straight line that intersects a maximum number of given vertical segments. Hence, the line transversal problem can also be solved optically in constant time.

4 Constructing a Bisector of a Finite Set of Points and Ham-Sandwich Cuts in Two Dimensions

A line L is called a *bisector* of a finite set P of points in $\Re^2$ if each open half-plane defined by L contains at most half of the points in P.

If we pass from the original plane to the dual plane, the set P becomes a set $D(P)$ of dual lines and a bisector L turns into a point $D(L)$ lying between the lower $\lfloor \frac{N}{2} \rfloor$-level and the lower $\lceil \frac{N}{2} \rceil$-level of the arrangement $A(D(P))$. Hence, we can propose the following algorithm for constructing a bisector:

Step 1. Dualize the given point set P into a set of dual half-planes $D^+(P)$.

Step 2. Threshold $D^+(P)$ to select all points at the $\lfloor \frac{N}{2} \rfloor$-level of intensity. As a result, we obtain dual points of all possible bisectors.

Step 3. Choose a point from the set obtained and dualize it into a bisector (see section 3.1 for the procedure of choosing a point from the set.)

A common bisector of two finite point sets (i.e. a Ham-sandwich cut of the sets) is constructed in a similar manner. First, we construct dual points of all possible bisectors separately for the first set and for the second set. Then we construct the intersection of these two sets of dual points and choose a point from the intersection. The point choosen is the dual point of a common bisector. Thus we obtain

Theorem 4.1 *A bisector of a finite point set in the plane, as well as a ham-sandwich cut of two finite point sets, can be constructed optically in constant time by a single optical processor which works in the monochromatic mode.*

5 Constructing a Center-point of a Point Set in the Plane

Recall that a centerpoint of a point set $P = \{P_i\}_{i=1}^{N}$ in the plane is a point p such that, for any straight line containing p, there are at least $\lceil \frac{N}{3} \rceil$ of the points of P in each closed half-plane determined by the line.

In the dual plane, a centerpoint corresponds to a line which lies between the upper $\lceil \frac{N}{3} \rceil$-level and the lower $\lceil \frac{N}{3} \rceil$-level of the arrangement of lines $D(P)$ obtained as a result of dualization of the point set P.

It is known that such a line always exists [E]. Moreover, we can select it in such a way that it passes through a pair of vertices of the above levels. It is also known that we are able to construct all such lines passing through a pair of points from a point set simultaneously in constant time [KS93]. Hence, the only thing that remains to do is to reject those lines which do not lie between the upper and lower $\lceil \frac{N}{3} \rceil$-levels.

Thus, constructing the dual line of a center-point of a point set in the plane can be performed optically as follows:

Step 1. Dualize points P into lines $D(P)$ and extract all their intersection points IP.

Step 2. Construct the upper and lower $\lceil \frac{N}{3} \rceil$-levels of $A(D(P))$ (in what follows we denote them as *levels*) and construct all their vertices V as follows:

$$V = levels \cap IP.$$

Step 3. Simultaneously construct all straight lines L passing through each pair of points of V.

Step 4. Fill the space between the levels constructed as follows:

$$space = Threshold_{\leq 2}(levels{+}ray\_up) \cup levels.$$

Step 5. Construct those segments S of lines L which do not lie between the above levels as follows: $S = L \setminus space$.

Step 6. Hough dualize the segments S into the points P_S which, in turn, Hough dualize back into straight lines L_{reject} supporting the segments S.

Step 7. Compute $L \setminus L_{reject}$. As a result we obtain the image $L_{remain} \setminus (L_{remain} \cap L_{reject})$, where L_{remain} are those lines of L which lie completely between the levels required.

Step 8. Using the algorithm described in [K92, KS92] dualize the image obtained into points P_{remain} of those lines which lie completely between the levels required.

Obviously, the latter points are sought for centerpoints.

Thus, we obtain

Theorem 5.1 *A centerpoint of a point set in the plane can be constructed optically in constant time by a single optical processor which works in the monochromatic mode.*

6 Finding a Separator Line for a Set of Segments

A line L is called a *separator* for a set S of objects in the plane if L avoids all the objects and partitions S into two nonempty subsets (see [ERS]). In this section we consider the problem of constructing optically a separator line for a set of segments.

In a dual setting, the problem appears to be as follows. The given segments are dualized into double wedges, and a separator line is dualized into a point lying in the complement of the union of these double wedges, strictly between their upper and lower envelopes [ERS]. Hence, we can propose the following optical algorithm for finding the set of all possible separators:

Step 1. Dualize all points of the segments into their dual lines. As a result, the segments are dualized into the union of double wedges $\bigcup_{e \in S} DW(e)$.

The upper (lower) envelope of the double wedges is the collection of all points of the lines bounding the double wedges, which are visible from $y = +\infty$ $(y = -\infty)$. Hence we can proceed as follows:

Step 2. Construct the upper and lower envelopes using the algorithm for hidden lines removal [K91, KS92].

Step 3. Construct the following Minkowski sums:

$$above\_upper\_envelope = upper\_envelope +$$
$$+ ray\_up;$$
$$below\_lower\_envelope = lower\_envelope +$$
$$+ ray\_down.$$

Step 4. Construct the set of dual points of all separators as follows:

$$P = \Re^2 \setminus \bigcup_{e \in S} DW(e)$$
$$\setminus above\_upper\_envelope \setminus below\_lower\_envelope.$$

Step 5. Choose a point from P and dualize it into a separator line.

Thus, we obtain

Theorem 6.1 *A separator line for a set of segments in the plane can be constructed optically in constant time by a single optical processor which works in monochromatic mode.*

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