The Delaunay triangulation maximizes the mean inradius

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Abstract

I prove that amongst all triangulations of a planar point set the Delaunay triangulation maximizes the arithmetic mean of the inradii of the triangles.

1 Introduction

A triangulation of a set of points is a partition of the convex hull into triangles. The *Delaunay triangulation* is a well known triangulation, being the planar dual of the famous *Voronoi diagram*.

Most applications of triangulations require that the triangulation should avoid 'skinny' triangles. Many different measures of the skinniness of a triangle have been proposed. One of these is the inradius (radius of the inscribed circle) [14, 19].

In this paper I prove that the Delaunay triangulation is the triangulation that maximizes the arithmetic mean of the inradius.

1.1 Background

Triangulating sets of points is a very important problem in computational geometry; there are far too many applications in computational geometry and other fields to mention here. (See the surveys [4, 6, 1])

There are many different possible triangulations of a set of sites. Which one is optimal will depend on the application. For example:

- If the triangulation is to be used as finite element mesh we wish to avoid ill-conditioned equations. This means that we do not want triangles with angles close to 180° [2].
- If we are using the triangulation to linearly interpolate functions with a bounded second derivative, then the error is minimized by minimizing the maximum circumradius of any triangle [15].
- If the triangulation represents a three-dimensional surface which is to be rendered on a raster display, then we want to avoid triangles less than one pixel wide as these can cause undesirable artifacts [8].

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Many other alternative definitions of optimality have been proposed. See [4, 13] for surveys.

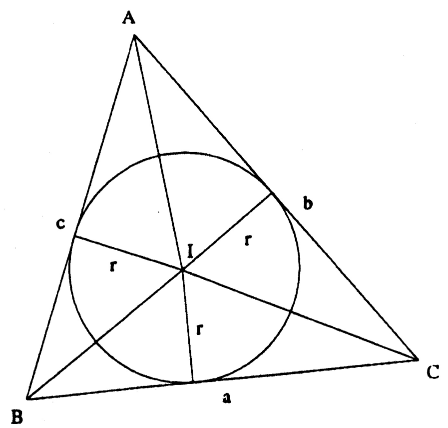
Amongst all triangulations, The Delaunay triangulation optimizes many triangulation measures. These include

- maximizing the minimum angle [20],
- minimizing the maximum circumscribed circle [5],
- minimizing the maximum smallest enclosing circle[1] [5, 17],
- minimizing the integral of the gradient squared [18, 16],

Little [14] and Schumaker [19] have proposed that the triangles in a good triangulation should have large inradii. I will prove that the Delaunay triangulation maximizes the sum of the inradii (and hence the arithmetic mean).

2 Preliminaries

I use $R(ABC)$ to denote the circumradius of the triangle ABC, $r(ABC)$ the inradius, $\Delta(ABC)$ the area, and $P(ABC)$ the perimeter. Note that $r(ABC) = 2\Delta(ABC)/P(ABC)$ (see figure 1).



$$\begin{aligned}\Delta(ABC) &= \Delta(IAB) + \Delta(IBC) + \Delta(ICA)\\ &= (rc + ra + rb)/2\\ &= rP(ABC)/2.\end{aligned}$$

Figure 1: $r(ABC) = 2\Delta(ABC)/P(ABC)$

A triangulation is *locally Delaunay* if for every pair of adjacent triangles ABC, ACD in the triangulation, the Delaunay triangulation of $ABCD$ includes the triangles

[1]The smallest enclosing circle differs from the circumscribing circle when the triangle is obtuse.

ABC and ACD. The key to the proofs that the Delaunay triangulation optimizes the measures listed above is the following fact: A triangulation is Delaunay iff it is locally Delaunay [9].

It follows from this fact that it is only necessary to prove that the Delaunay triangulation optimizes a measure for sets of four points, since if the triangulation that optimizes that measure is not Delaunay, then that triangulation is not locally Delaunay. Therefore there must be an adjacent pair of triangles ABC, ACD in the triangulation such that the Delaunay triangulation of $ABCD$ is ABD, BCD. If the Delaunay triangulation optimizes that measure for sets of four points, then we can find a triangulation with a better measure by replacing the triangles ABC, ACD with ABD and BCD, which is a contradiction.

3 Main result

Theorem 1. *The Delaunay triangulation maximizes the sum of the inradii.*

PROOF. It is sufficient to prove that if $ABCD$ is a convex quadrilateral with Delaunay triangulation ABC, ACD then $r(ABC) + r(ACD) > r(ABD) + r(BCD)$.

Let P be the point where the diagonals of $ABCD$ intersect. Let $r_A = r(DAB)$, $r_{AB} = r(PAB)$, $r_{DA} = r(PDA)$ and h_A be the length of the altitude at A in triangle DAB (see figure 2).

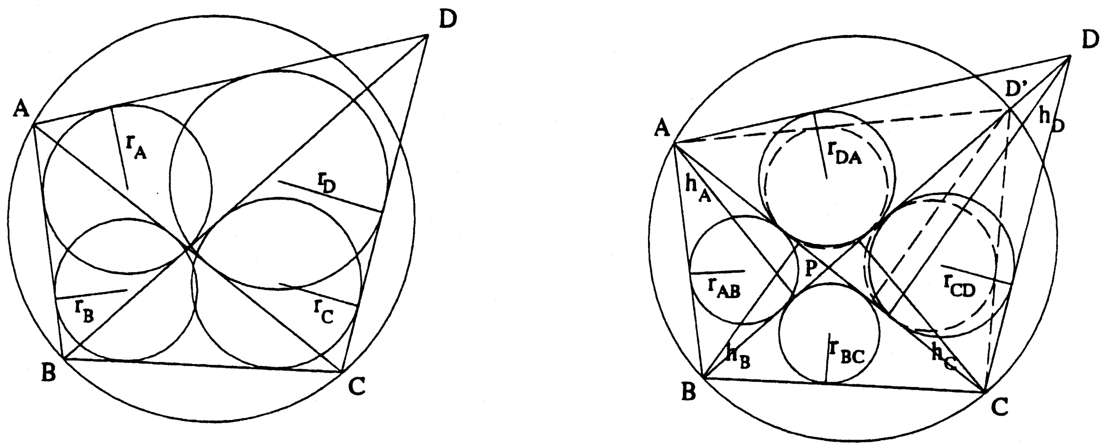


Figure 2: Inscribed Circles

Demir [7] has proved the following relation between these quantities:

$$r_{DA} + r_{AB} - r_A = 2\frac{r_{DA}r_{AB}}{h_A}. \tag{1}$$

Applying this relation to triangles ABC, BCD, and CDA yields:

$$r_{AB} + r_{BC} - r_B = 2\frac{r_{AB}r_{BC}}{h_B}, \tag{2}$$

$$
\begin{aligned}
r_{BC} + r_{CD} - r_C &= 2\frac{r_{BC}r_{CD}}{h_C}, & (3)\\
r_{CD} + r_{DA} - r_D &= 2\frac{r_{CD}r_{DA}}{h_D}. & (4)
\end{aligned}
$$

Adding 1 and 3 and subtracting 2 and 4 we get:

$$
r_B + r_D - r_A - r_C = 2\left(\frac{r_{DA}r_{AB}}{h_A} - \frac{r_{AB}r_{BC}}{h_B} + \frac{r_{BC}r_{CD}}{h_C} - \frac{r_{CD}r_{DA}}{h_D}\right). \quad (5)
$$

Now let D' be the point where BD intersects the circumcircle of ABC, and define $r'DA$, $r'CD$ and h'_D appropriately.

$PD'A$ is similar to PCB; so

$$
\frac{r'_{DA}}{h_A} = \frac{r_{BC}}{h_B} \quad \text{and} \quad \frac{r_{BC}}{h_C} - \frac{r'_{DA}}{h'_D}.
$$

Clearly $r'_{DA} < r_{DA}$. Also,

$$
\begin{aligned}
\frac{r_{DA}}{h_D} &= \frac{2\Delta(PDA)}{h_D(|PD| + |DA| + |AP|)}\\
&= \frac{|AP|}{(|PD| + |DA| + |AP|)}\\
&< \frac{|AP|}{(|PD'| + |D'A| + |AP|)}\\
&= \frac{r'_{DA}}{h'_D}.
\end{aligned}
$$

Using these results in equation 5 we get

$$
r_B + r_D - r_A - r_C > 2\left(r_{AB}(\frac{r'_{DA}}{h_A} - \frac{r_{BC}}{h_B}) + r_{CD}(\frac{r_{BC}}{h_C} - \frac{r'_{DA}}{h'_D})\right) = 0.
$$

That is, $r(ABC) + r(ACD) > r(ABD) + r(BCD)$. □

Remark

As an immediate consequence of thereom 1 we have:

Theorem 2. *$r(ABC) + r(CDA) = r(DAB) + r(BCD)$ if and only if $ABCD$ is cyclic.*

The first known statement of the "if" part of this theorem was on a tablet hung in a Japanese temple in 1800 [10]. It is the most celebrated Japanese temple geometry theorem, mentioned or proved in [21, 11, 10, 12]. None of these proofs can be easily modified to prove the converse; so the "only if" part is a new result in elementary geometry.

Remark

This result does not generalize to three dimensions as the following counterexample shows.

The points $A = (0,0,0)$, $B = (1,0,0)$, $C = (0,1,0)$, $D = (0,0,1)$, $E = (1,1,1)$ lie on a common sphere. The convex polyhedron $ABCDE$ can be divided into tetrahedra in two ways: the two tetrahedra $ABCD$ and $BCDE$, or the three tetrahedra $AEBC$, $AECD$, and $AEDB$.

tetrahedron	volume	area	inradius
$ABCD$	1/6	$(3+\sqrt{3})/2$	$\frac{1}{2}-\frac{1}{6}\sqrt{3}$
$BCDE$	1/3	$2\sqrt{3}$	$\frac{1}{6}\sqrt{3}$
$AEBC$	1/6	$(1+2\sqrt{2}+\sqrt{3})/2$	$1/(1+2\sqrt{2}+\sqrt{3})$
$AECD$	1/6	$(1+2\sqrt{2}+\sqrt{3})/2$	$1/(1+2\sqrt{2}+\sqrt{3})$
$AEDB$	1/6	$(1+2\sqrt{2}+\sqrt{3})/2$	$1/(1+2\sqrt{2}+\sqrt{3})$

Clearly $r(ABCD) + r(BCDE) \neq r(AEBC) + r(AECD) + r(AEDB)$.

Remark

The result leads to an alternative implementation of the `InCircle` geometric primitive, required for the construction of the Delaunay triangulations. Unfortunately, because six sqaure root operations are required to calculate the four inradii, this alternative implementation is slower than the usual determinant based one.

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