On Delaunay Oriented Matroids.

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Abstract: The fact that for any finite set S of points in the Euclidean plane E^2 one can define an oriented matroid in terms of how spheres partition it is well-known and easy to proof via the lifting property of Delaunay triangulations (cf. [1] and [2]) . Here we give a new definition of these oriented matroids (that we call *Delaunay oriented matroids*) which trivially generalizes to arbitrary dimension and explicitly show the precise relation of Delaunay oriented matroids with not only the usual Voronoi diagrams and Delaunay triangulations, but with Voronoi diagrams of arbitrary order k.

Moreover we show that the existence of these Delaunay oriented matroids is not really dependent on the lifting property of Delaunay triangulations but on some nice properties of Euclidean spheres. In fact, we generalize the definition to smooth, strictly convex distances in the plane (cf. §2) which have not the lifting property as we also show and to any metrics whose "spheres" have some nice properties (cf. §3). The difference is that, in these cases, the resulting oriented matroids can be non-realizable.

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1. The Euclidean case.

One of the fields where the theory of oriented matroids has an application is in Voronoi diagrams and their dual Delaunay triangulations. If S is a finite set of points (to be called *sites*) in the Euclidean d-space E^d, an oriented matroid can be constructed that contains the vicinity information between points of S in much the same way that Delaunay triangulations do. For this reason we call them *Delaunay oriented matroids.*

These oriented matroids where first introduced by Bland and Las Vergnas [2] (see also [1], where their relation with Delaunay triangulations is explicitly mentioned) but there only the two dimensional case is considered. In this section we give a general definition for arbitrary dimension and study their main properties.

An *oriented matroid* in N points can be defined as a collection $\mathcal{M}$ of signed partitions (C^+, C^0, C^-), called *covectors*, of a finite set S of cardinality N with the collection $\mathcal{M}$ verifying some conditions. These conditions (or *axioms*) were devised to abstract how linear or affine hyperplanes partition a finite set of points in the real vector space $\mathbb{R}^d$ or in the Euclidean space E^d. Nevertheless some oriented matroids exist which cannot be *realized* in this way, the so-called *non-realizable* or *non-representable* oriented matroids.

In the oriented matroids that we are going to consider, S is a finite set of points in the Euclidean space E^d and covectors describe how S is partitioned by Euclidean spheres and hyperplanes, where a sphere is meant to be a scaled translation of the unit sphere. More explicitly:

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Definition 1.1 Let S be a finite set of points in E^d. A signed partition $C = (C_-, C_0, C_+)$ is a *covector* of the *Delaunay oriented matroid* DOM(S) of S if either $C = (C_-, C_0, C_+) = (\emptyset, S, \emptyset)$, or there exists a sphere or a hyperplane C such that $C_0 = C \cap S$ and C_- and C_+ lie respectively in the two connected components of $E^d \setminus C$.

For this definition to make sense one must verify that the described set of covectors actually satisfies the axioms for an oriented matroid. This can be made in an elegant way via an indirect construction similar to the one used by Brown ([3]) to derive Delaunay triangulations from convex hulls of a lifted set of sites. Consider the lifting map $f : E^d \to E^{d+1}$ defined by $f(x_1, \ldots, x_d) = (x_1, \ldots, x_d, \sum x_i^2)$ and call $\tilde{S}$ the image of S by f. Note that $f(E^d)$ is a paraboloidal hypersurface in E^{d+1} whose intersection with any hyperplane projects down onto a sphere or a hyperplane in E^d. From this it is easy to deduce that the covectors of DOM(S) lift to the signed partitions of $\tilde{S}$ obtained by hyperplanes in E^{d+1}, and thus that DOM(S) coincides with the usual *hyperplane oriented matroid* defined from $\tilde{S}$.

Incidentally, the fact that Euclidean Delaunay triangulations can be derived from lower envelopes of some lifted sites via the above construction will be referred to as the *lifting property* of the Euclidean distance. The lifting property also shows that DOM(S) is a *realizable* oriented matroid. Some additional properties of DOM(S) are summed in the following statement.

Proposition 1.2 *Let S be a finite set of points in E^d and let k be the dimension of the affine subspace spanned by S. Then the Delaunay oriented matroid* DOM(S) *of S is acyclic, polytopal and realizable. Its rank equals $k+1$ if S is contained in a sphere and $k+2$ otherwise.*

The definition of the Delaunay oriented matroid of S in terms of spheres suggests its connection with the Voronoi diagram of S and its dual the Delaunay diagram. We use the term *Delaunay diagram* to refer to the topological dual of the Voronoi diagram (as opposed to the term *Delaunay triangulation* which is not well defined if $d+2$ points of S are co-spherical). The Delaunay diagram of S is a polyhedral complex in E^d whose cells are the convex hulls $< T >$ of those subsets T of S for which a sphere exists passing through all the points in T and having $S \setminus T$ outside. Then, for any set $T \subset S$ whose convex hull is a cell in the Delaunay diagram of S, the Delaunay oriented matroid DOM(S) must contain a covector $(\emptyset, T, S \setminus T)$.

Unfortunately the converse is not true and thus the Delaunay oriented matroid does not contain all the information to recover the Delaunay diagram. In fact, a covector as described above associated to a subset T can be produced by a sphere passing through T but having $S \setminus T$ inside. Technically speaking we can say that the Delaunay oriented matroid DOM(S) completely describes the *Delaunay complex* as introduced in [6], but not the Delaunay diagram. We recall that the Delaunay complex is the convex hull of the lifted set $\tilde{S}$ in E^{d+1} that appears in the lifting property. Its lower part projects down onto the Delaunay diagram and its upper part projects onto the *furthest site Delaunay triangulation.*

An oriented matroid containing *all* the combinatorial information of the Delaunay diagram can be obtained from the Delaunay oriented matroid by just introducing a new point ∞ which is considered to lie in any hyperplane and in the exterior region of any sphere. The signed partitions of $S \cup \{\infty\}$ induced by all spheres and hyperplanes are again the covectors of an oriented matroid that we call the *extended Delaunay oriented matroid* EDOM(S) of S. It has almost the same properties as the Delaunay oriented matroid.

Proposition 1.3 *Let S be a finite set of points in E^d and let k be the dimension of the affine subspace spanned by S. Then the extended Delaunay oriented matroid* EDOM(S) *of S is acyclic, polytopal and realizable and has rank $k+2$. Its deletion at ∞ is the Delaunay oriented matroid* DOM(S) *of S and its contraction at ∞ is the usual affine oriented matroid of S.*

This new oriented matroid EDOM(S) contains the combinatorial information of the Voronoi and Delaunay diagrams, in the following sense.

Proposition 1.4 *Let S be a finite set of points in E^d and consider their extended Delaunay oriented matroid* EDOM(S). *Then,*

i) The Voronoi regions of two points $A, B \in S$ are adjacent (i.e, their boundaries intersect) if and only if EDOM(S) *contains a covector (C_+, C_0, C_-) with $C_+ = \emptyset$ and $A, B \in C_0$.*

ii) The convex hull $< T >$ of a subset $T \subset S$ is a cell in the Delaunay diagram of S if and only if EDOM(S) *contains the covector $(\emptyset, T, S \cup \{\infty\} \setminus T)$.*

Even more, EDOM(S) contains also the combinatorial information for the higher order Voronoi diagrams as defined by Shamos and Hoey ([12]). For a set S of sites, the order k Voronoi diagram k-$Vor(S)$ partitions the plane in regions having the same k nearest points among S. Thus, a subset $T \subset S$ of cardinality k defines a nonempty region in k-$Vor(S)$ if and only if a sphere exists containing T and having $S \setminus T$ outside. Thus:

Proposition 1.5

i) A subset T of cardinality k defines a non-empty region k-$Reg(T)$ in k-$Vor(S)$ if and only if the signed partition $(T, \emptyset, (S \cup \infty) \setminus T)$ is a covector in EDOM(S).

ii) A collection k-$Reg(T_1), \ldots, k$-$Reg(T_t)$ of such non-empty regions in k-$Vor(S)$ have a common boundary point (i.e. are adjacent) if and only if the signed partition $(T_\cap, T_\cup \setminus T_\cap, (S \cup \{\infty\}) \setminus T_\cup)$ is a covector in EDOM(S), *where $T_\cup = \cup T_i$ and $T_\cap = \cap T_i$.*

2. The case of convex distance functions

The study of Voronoi diagrams for metrics other than the Euclidean one is of interest in many applications. See, for example, [8] for some developments in this area. The class of metrics that we are going to study in this section are the so-called *convex distance functions*. Convex distance functions were introduced in the context of Voronoi diagrams by Chew and Drysdale [4] who gave an algorithm for computing the Voronoi diagrams produced by them and some of their applications. See also [5], [7] and [10] for some properties of convex distance functions.

Let K be a convex body in the Euclidean space E^d containing the origin in its strict interior. We can use K to measure "distances" in E^d in the following way. Let A and B be two arbitrary points in E^d. The K-distance $\delta_K(A, B)$ from A to B is the only scaling factor λ that makes the scaled translation $K + \lambda A$ have B on its boundary. The distance function δ_K so defined is called a *convex distance function.* If K is centrally symmetric then δ_K is actually a Minkowski metrics.

An alternative (axiomatic) definition stressing their properties is:

Definition 2.1 A *convex distance function* in the Euclidean d-space E^d ($d \geq 2$) is a map $\delta : E^d \times E^d \longrightarrow \mathbb{R}_{\geq 0}$ satisfying the following properties.

(D0) $\delta(A, B) \geq 0 \quad \forall A, B \in E^d$ and $\delta(A, B) = 0$ iff $A = B$.

(D1) $\delta(A, B) + \delta(B, C) \geq \delta(A, C) \quad \forall A, B, C \in E^d$ (*triangle inequality*).

(D2) $\delta(A, B) = \delta(A + v, B + v) \quad \forall A, B \in E^d, \forall v \in \mathbb{R}^d$ (*invariance under translations*).

(D3) $\delta(A, B) + \delta(B, C) = \delta(A, C) \quad \forall B \in [A, C]$, where $[A, C]$ denotes the line segment between A and C (*additivity on segments*).

Additional conditions on the defining convex K give additional properties of the convex distance:

i) If K is strictly convex, (i.e. contains no line segments on its boundary) then the triangle inequality is strict: $\delta_K(A, B) + \delta_K(B, C) = \delta_K(A, C)$ iff $B \in [A, C]$. We say in this case that δ_K is *strictly convex.*

ii) If δ_K is smooth (i.e. it has a unique supporting hyperplane at every boundary point) then every $d+1$ points not lying in any hyperplane are δ_K-cospherical (the proof of this can be found in [9]). We say then that δ_K is *smooth.*

Reciprocals of both are easy to proof. If K is not strictly convex or not smooth one can find violations of the strict triangle inequality, or $d+1$ points neither co-hyperplanar nor δ-cospherical. In the plane, property (ii) is even stronger. Only one δ_K-sphere passes through any three given non-collinear points. This makes Voronoi and Delaunay diagrams for smooth, strictly convex distances in the plane very similar to Euclidean ones, but does not hold in higher dimensions (see [7]).

Moreover, as a result of these nice properties, smooth, strictly convex distances in the plane produce Delaunay oriented matroids in the same way as the Euclidean distance does, except for their realizability (which in the Euclidean case comes from the lifting property of Euclidean Delaunay triangulations). Proofs of results 2.2–2.5 will be given in [11].

Proposition 2.2 *Let δ_K be a smooth, strictly convex distance function in E^2. For any set S of sites call* $\mathrm{DOM}_\delta(S)$ *the collection of signed partitions of S defined by all hyperplanes and δ_K-spheres. Then* $\mathrm{DOM}_\delta(S)$ *is an oriented matroid for all possible sites S if and only if K is strictly convex and smooth. In this case the extension of this oriented matroid to the infinity point is again an oriented matroid* $\mathrm{EDOM}_\delta(S)$.

Proposition 2.3 *All the properties described in Propositions 1.2–1.5 for Delaunay oriented matroids with the Euclidean distance are still true for smooth, strictly convex distances in the plane, except for their realizability.*

Concerning realizability we have the following negative result, very similar in its formulation to the negative Theorem 3 of [5]. The condition of K being centrally symmetric in Proposition 2.4 is a technical one used in the proof but probably not needed in the statement. In the contrary, the fact of K not being an ellipse is needed because otherwise δ_K is an affine transform of the Euclidean distance and it can only produce Delunay diagrams and oriented matroids equivalent to Euclidean ones.

Proposition 2.4 *Let K be a symmetric smooth strictly convex body in the plane, but not an ellipse. Then, the Delaunay oriented matroid of some set S of eight points with respect to δ_K is not realizable.*

The proof of Proposition 2.4 consists on finding eight points whose Delaunay diagram contains the eight triangles shown in figure 1, and such that points $\{1,3,5,7\}$ and $\{2,4,6,8\}$ are collinear. This two facts prevent the corresponding Delaunay oriented matroid from being realizable. Let us mention that our 8-points example is minimal because any oriented matroid in less then 8 points is realizable.

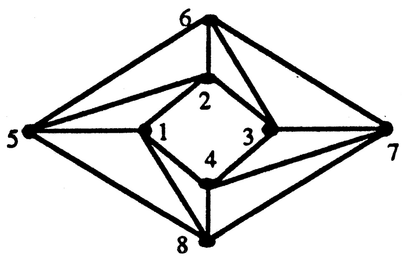

Figure 1

An interesting corollary comes from the fact that such a diagram is not *regular* (i.e. cannot be the projection of the lower envelope of any lifting of the eight points into 3-space). Thus, Delaunay triangulations for these distances cannot be derived from convex hulls in the way of [3].

Corollary 2.5 *The only (symmetric) convex distance functions in the plane which have the lifting property are affine transforms of the Euclidean distance.*

3. A more abstract setting.

The purpose of this section is to stress how the fact that smooth, strictly convex distance functions in the plane produce Delaunay oriented matroids comes from nice geometrical properties between their δ_K-spheres. We are going to abstract these properties and generalize them (in Definition 3.1) in such a way that they still permit to obtain Delaunay oriented matroids from spheres.

We will work in the compactified space $E^d \cup \{\infty\}$ homeomorphic to a sphere S^d in order to unify the description of hyperplanes and spheres. A tamely embedded topological sphere S^k ($k = 0 \dots d$) in S^d is said to be a *pseudo-sphere*. Our idea is to cover the sphere S^d by a family of pseudo-spheres S^{d-1} with the same combinatorial properties that the Euclidean spheres and hyperplanes, where the role of hyperplanes will be played by pseudo-spheres passing through the distinguished infinity point, although this point has no special role in the axioms we pose for our pseudo-spheres. These axioms are:

Definition 3.1 A *Delaunay system of spheres* in S^d ($d \geq 2$) is a collection $\mathcal{S}$ of dimension $d-1$ pseudo-spheres in S^d (to be called $\mathcal{S}$*-spheres*) satisfying the following axioms.

(S1) Through any $d+1$ points in S^d there passes at least one $\mathcal{S}$-sphere.

(S2) Two any non disjoint $\mathcal{S}$-spheres C and D intersect either in a $(d-2)$-dimensional tame sphere or in a point. We say that they intersect *transversally* and *tangentially*, respectively.

(S3) For any $\mathcal{S}$-sphere C, the collection of $(d-2)$-dimensional spheres $\mathcal{S}_C := \{C \cap D \mid D \in \mathcal{S}, C \text{ and } D \text{ intersect transversally}\}$ is a Delaunay system of spheres in C.

(S4) For any two points P and Q and any $\mathcal{S}$-sphere C passing through P but not through Q there exists one only $\mathcal{S}$-sphere D passing through Q and intersecting C tangentially at P.

(S5) For any two disjoint $\mathcal{S}$-spheres C and D and any point p "between" them (i.e. in the component of $S^d \setminus C$ which contains D and in the component of $S^d \setminus D$ which contains C) there exists an $\mathcal{S}$-sphere Z passing through p and separating C and D (i.e. with C and D respectively contained in the two components of $S^d \setminus Z$).

Remarks:

- the definition is inductive on the dimension d, because of axiom (S3). For $d = 1$ we define the collection $\mathcal{S}_1 = \{\{P, Q\} \mid P, Q \in S^1\}$ to be the only Delaunay system of spheres in S^1. In this case we will say that two $\mathcal{S}_1$-spheres C and D intersect tangentially if they have a common point, that they "intersect" transversally if they do not have any common point but each component of $S^1 \setminus C$ contains one of the points of D and that they do not intersect otherwise. With these definitions $\mathcal{S}_1$ satisfies axioms (S1), (S2), (S4) and (S5). The inductive axiom (S3) is somehow satisfied if we consider $\mathcal{S}_0 = \{\emptyset\}$ to be the only possible matroidal system of spheres in $S^0 = \{-1, +1\}$, with the usual convention $S^{-1} = \emptyset$.

- axioms (S4) and (S5) hold for smooth, strictly convex distances in any dimension and probably for any metrics whose spheres are smooth, strictly convex hypersurfaces, not necessarily homothetical to one another.

- in the planar case ($d = 2$), axioms (S1) and (S2) are equivalent to "through any three points there passes one and only one $\mathcal{S}$-sphere" and axiom (S3) is redundant. Thus, they they hold for smooth strictly convex distances. In higher dimension axioms (S2) and (S3) imply that through any $d+1$ points there passes at most one $\mathcal{S}$-sphere and they do not hold, in general, for smooth, strictly convex distances (because of the results in [7]).

Our main result in this section says that whenever a metrics has spheres satisfying the axioms above (in the compactified S^d, then Delaunay oriented matroids can be defined as those for the Euclidean distance. For the last part of the statement to make sense we need the fact that an arbitrary intersection of $\mathcal{S}$-spheres satisfying our axioms is a pseudo-sphere and thus has a well-defined dimension.

Proposition 3.2 *Let $\mathcal{S}$ be a Delaunay system of spheres in S^d and let $P = \{1, .., N\}$ represent N points in S^d. Consider the collection $\mathcal{V}$ of signed subsets of P consisting on the empty signed set $(\emptyset, P, \emptyset)$ and those which are respectively positive and negative in the two sides of a certain $\mathcal{S}$-sphere. Then, $\mathcal{V}$ is the set of covectors of a certain oriented matroid $\mathcal{M}$, which is acyclic and polytopal. This oriented matroid has rank equal to 2 plus the dimension of the intersection of all $\mathcal{S}$-spheres containing S.*

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