Some Qualitative Properties of a Generalized Voronoi Diagram for Convex Polyhedra in d-dimensions[1]

Abhi Dattasharma
Department of Computer Science and Automation
Indian Institute of Science, Bangalore 560 012, India
e-mail: abhi@chanakya.csa.iisc.ernet.in

1. Introduction

Generalized Voronoi diagrams are extensions of classical Voronoi diagrams[1] in general dimensions and for general sets. These diagrams are geometric constructs which are interesting in their own right; they also prove to be very useful in applications. It is quite clear that for such a diagram to be used efficiently, a comprehensive knowledge about its geometry is necessary. For example, if such a diagram is to be used in motion planning problems, it is absolutely necessary to know about its connectivity; and also about the properties of its disconnections, if any. Such examples assert that a thorough knowledge of the qualitatitve properties for such a diagram is useful in practical applications besides being in itself an interesting geometric problem.

In this paper, we establish some qualitative properties for a generalized Voronoi diagram for convex polyhedra in d-dimensions. This generalized Voronoi diagram was proposed for three dimensions in [2,3] for a convex polyhedron M with non-empty interior moving among convex, pairwise interior disjoint polyhedral obstacles O_is with non-empty interiors and certain qualitative properties for the diagram were established. This paper carries the idea to general dimensions; it shows that even in d-dimensions the diagram permits a nice structure: in fact one important result remains entirely unaltered. We believe that this is the first attempt to establish such nice geometric and qualitative properties of a generalized Voronoi diagram for convex polyhedra in d-dimensions.

2. Preliminaries

In this section, we briefly describe some important definitions. More details can be found in [4].

Definition 2.1 A set S is said to be *polyhedral* if S can be written as a finite union of convex polyhedra, i.e., $S = \bigcup_{i=1}^{n} P_i$, where each P_i is a convex polyhedron and n is finite.

Definition 2.2 Suppose X_i, $i = 1, \ldots, n$ are polyhedral sets. Let E_i be the set of all open 1-faces of X_i and V_i be the set of all 0-faces of X_i. Then we call the set $S = \bigcup\{E_i \cup V_i\}$ the *wireframe* of $\{X_i\}$.

The following definition is taken from a paper by Leven and Sharir[5].

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Definition 2.3 Let $x \in R^d$. The *M-distance of a set A from x* is defined as the minimum expansion required of M when v_{ref} is placed at x such that M "just touches" A. Formally,

$$d(x; A) = \inf\{\lambda : (x + \lambda M) \bigcap A \neq \phi, \lambda \geq 0\}$$

If $x \in A$ then $d(x; A) = 0$. For convenience, we write $d(x; O_i)$ as $d_i(x)$.

Definition 2.4 Let O_i be an obstacle. Then *the cell associated with* O_i, C_i is the set

$$\{x \in R^d : d_i(x) \leq d_j(x) \ \ \forall j \neq i, j \in 1, \ldots, Q\}$$

Physically, this is the set of all points in R^d from where M is closer to O_i than any other obstacle. It is not difficult to see that each cell is polyhedral.

Definition 2.5 Let $x \in R^d \setminus \bigcup O_i$ and consider $(x + d_i(x)M)$. Clearly $(x + d_i(x)M)$ touches O_i. Then, by convexity of O_i and M, a unique open facet o_i of O_i is being touched by a unique open facet o_m of M. We call the ordered pair (o_i, o_m) as the *touch description associated with the touch.*

Definition 2.6 For the touch description associated with a touch t we define *loss of degrees of freedom* $ldof(t)$ as $(d + 1)-$ the number of free variables in the set of linear equations which describe the touch.

Consider an $x \in R^d$. Suppose x is such that exactly k obstacles $O_1, \ldots, O_k$ are equidistant from x and no other obstacle is as close as any of these k obstacles. Then when M is placed with v_{ref} on x and expanded by $d_1(x)$, there are exactly k touches, say $t_1, \ldots, t_k$. Each one of these touches t_i has one touch description (o_i, o_{m_i}) associated with it.

Definition 2.7 We call the list $(o_1, o_{m_1}, \ldots, o_k, o_{m_k})$ as the *type of touch* T at x.

Definition 2.8 Consider a type of touch T. By definition T is a $2k$-tuple $(o_1, o_{m_1}, \ldots, o_k, o_{m_k})$. The *loss of degrees of freedom associated with the type of touch* T, $ldof(T)$ is defined as the sum of the loss of degrees of freedom for each touch description (o_i, o_{m_i}) associated with the touch t_i, i.e. $ldof(T) = \sum ldof(t_i)$.

As in [3], we make certain generic assumptions[6] on the relative orientations of the obstacles. We give here only one of those which we will require later.

Assumption Let $k \geq 1$ and consider any k distinct touches, each touch being described by a touch description $t_i, i = 1, \ldots, k$. Then the set of all points where exactly these k touches (and no other) are maintained is either empty or a $(d+1) - (\sum ldof(t_i))$ dimensional manifold. A set having negative dimension is taken as the null set.

We refer to this assumption as independence[7].

We use the name *skeleton* for the Voronoi diagram which is formally defined as:

Definition 2.9 The *skeleton of* $R^d \setminus \bigcup O_i$ is the wireframe of $\{C_i\}_{i=1}^{i=Q}$ where C_i is as in definition 2.4 and wireframe is as in definition 2.2.

3. Qualitative Properties

In this section, we give the main results. Because of lack of space, we omit all lengthy proofs.

Full details can be found in [4]. Also, we believe an understanding of the results for three dimensions[3] will help the reader.

Proposition 3.1 In general, the skeleton may have several disconnected components in one connected component of the free space.

Proof See [7] for an example which proves this result for three dimensions. ∎

Definition 3.1 An obstacle O_i is said to be *active at a point* x if $d_i(x) \leq d_k(x)\ \forall k \in \{1, 2, \ldots, Q\}$. An obstacle O_i is said to be *active in a set* A if O_i is active at $x\ \forall x \in A$.

In the following, we will use the word "polytope" as a short form for "polytope lying on the union of the boundaries of the cells".

Definition 3.2 Suppose P_1 and P_2 are k dimensional polytopes such that the following hold: i) $P_1 \subset relint(P_2)$ and ii) there exists open set O, $O \supset P_1$ such that the type of touches remain unchanged at $x\ \forall x \in relint(O \bigcap P_2) \setminus P_1$. Then we say: P_1 is a *contained polytope*, P_2 is a *container polytope*, P_2 *contains* P_1, and P_1 is *contained* by P_2.

Proposition 3.2 Let P_1 and P_2 be two contained polytopes. Then $P_1 \bigcap P_2 = \phi$.

Proof If P_1 and P_2 are contained in two different container polytopes then the result is trivial. So suppose both P_1 and P_2 are contained in the same polytope P. Then P_1, P_2 and P are of the same dimension, say k.

Suppose $P_1 \bigcap P_2 \neq \phi$. Suppose $P_1 \bigcap P_2$ is a facet of dimension m, $k-1 \geq m \geq 0$. Consider the touches T_1 associated with P_1 for any $x \in P_1 \bigcap P_2$. Then $ldof(T_1) = (d - m + 1)$. Again, consider the touch T_2 associated with P_2 for any $x \in P_1 \bigcap P_2$. Then $ldof(T_2) = (d - m + 1)$. But at least one touch pair in T_1 is distinct from all touch pairs in T_2 as $P_1 \neq P_2$ and both of these are contained. This implies that total loss of degrees of freedom at any $x \in P_1 \bigcap P_2$ is at least $(d - m + 2)$, which is a contradiction. ∎

Definition 3.3 Every path component of the skeleton is called a *skeleton component*.

Definition 3.4 Suppose P is a polytope and S_1 is a skeleton component such that $S_1 \bigcap bd(P) \neq \phi$. Then S_1 is said to be the *skeleton component associated with* P.

The structural results, which we call "structural lemmas" require several subresults for their proof. Because of lack of space we don't include all those results and their proofs; instead we try to give some intuitive justification towards these lemmas. We will discuss the case of three dimensions as that is intuitively easy to follow. Full details can be found in [4].

Consider a polygon P on the boundary of a cell which contains other polygons inside it. Since we are considering three dimensions, there are two obstacles which are equidistant from P. For every point x in the interior of P (with all its contained polygons removed) we can find the closest points on these two obstacles. Then it is intuitively clear that no point on the line segments joining x and these two closest points can be a skeleton point. The following proposition states this result formally in d-dimensions.

Proposition 3.3 Let P be a container polytope and $\{P_1, \ldots, P_r\}$ be the set of all polytopes contained by P. Let $U = \{x : x \in relint(P \setminus \bigcup P_k)\}$. Suppose the obstacles active in

U are $O_1, \dots, O_p$. Let $V = \{\bigcup_{i=1}^{p} \overline{xy_{ix}}\ \forall x \in U\}$. Then V does not have a skeleton point.

Proof See [3] for the proof for three dimensions. A similar proof holds for general dimensions. ∎

Given the above proposition, it is easy to "see" the following. Suppose P contains several polygons inside it. Then the set V (as in proposition 3.3) "encloses" all the skeleton components associated with these contained polygons, and since these contained polygons are disjoint (proposition 3.2) there is a part of V separating every pair of these components. This leads us to the following result.

Lemma 3.1 Let P be a container polytope of dimension g and $\{P_1, \dots, P_r\}$ be the set of all polytopes of dimension g contained by P. Then the skeleton components associated with P and P_k are disjoint $\forall k = 1, \dots, r$. Also, for $m, s \in \{1, \dots, r\}$ and $m \neq s$, the skeleton components associated with P_m and P_s are disjoint.

Since the cells are connected, it is clear that if there are two skeleton components in one connected component of the free space then they are "neighbouring components", i.e., both of them has parts common to one cell, and therefore one of them is associated with a contained polygon of the cell whereas the other is associated with the container polygon. This result extends over any number of components and over any dimensions: it is formally stated as follows.

Lemma 3.2 Suppose there exist m skeleton components $S_1, \dots, S_m$ in one connected component of free space, and $m > 1$. Then each skeleton component S_i either has a polytope P_i which contains a polytope P_j of some other skeleton component $S_j, j \neq i$ or has a polytope P_i which is contained in a polytope P_j of some other skeleton component $S_j, j \neq i$.

Consider a polygon P in three dimensions containing a polygon P_1. Since we are considering three dimensions, and P_1 is contained in the relative interior of a polygon P there exists an obstacle O which is associated with P_1 and not P. This is easy to see: P has exactly two obstacles active in its relative interior with a loss of degrees of freedom = 2 whereas the boundary of P_1 needs a loss of degrees of freedom ¿ 2. Thus the cell associated with the obstacle O is wholly "enclosed" within the set V formed by the relative interior of P (with P_1 removed. see proposition 3.3). This immediately tells us that O can never contribute a point to any skeleton component lying "beyond" V. This result nicely generalizes as the following lemma.

Lemma 3.3 Let S_l be a skeleton component. Let p be any point on S_l and $O_1, O_2, \dots, O_r$ be the obstacles active at p. Then there does not exist a skeleton component S_m such that $S_m \bigcap S_l = \phi$, and S_m has a point q at which $O_1, O_2, \dots, O_r$ are active.

By the discussion preceding lemma 3.2, the following result is easy to see.

Lemma 3.4 Suppose P and P_1 are two polytopes in the boundary of cell C_i. If the skeleton component associated with P and the skeleton component associated with P_1 are disjoint then at least one of P and P_1 is a contained polytope.

By the discussion preceding lemma 3.3, we see that the cell associated with O lies wholly "within" V. Now consider the skeleton component associated with the contained polygon P_1. If any of the polygons (not P_1) which are connected to this skeleton component is a contained polygon then we can use the same argument, and prove the following result.

Lemma 3.5 Let S be a skeleton component. Let P be a polytope such that the skeleton of $P \subset S$ and P is contained by another polytope P_1. Then there does not exist any polytope P_2 such that $P_2 \neq P$, skeleton of $P_2 \subset S$ and P_2 is contained.

Also the following very interesting result is now intuitively easy; each contained polygon has one obstacle associated with it which is wholly enclosed locally, and the generalization is:

Lemma 3.6 There exist only $O(Q)$ contained polytopes.

Using the above lemmas, it can be established that:

Lemma 3.7 Among all the skeleton components in one connected component of the free space, exactly one of them has all non-contained polytopes.

These results characterize the generalized Voronoi diagram in a geometric way. Note that all the results except lemma 3.6 are geometric in nature. Also, there are remarkable similarities between the corresponding results in three dimensions[3] and the results in d-dimensions.

Lemma 3.6 needs special mention. The result for three dimensions has exactly the same statement; this shows that the number of "disconnections" in the generalized Voronoi diagram is *independent of the dimension as well as the size of the moving object and obstacles.* This itself is a very interesting topological result. This shows that if the number of obstacles is assumed to be a constant then the generalized Voronoi diagram in any dimension can always be made complete (i.e., having only one connected component in one connected component of the free space) by addition of a constant number of extra edges. This clearly tells us that this diagram is very suitable for application in motion planning problem for convex polyhedra.

4. Conclusion

In this paper, we stated some qualitative properties of a generalized Voronoi diagram for convex polyhedra in d-dimensions. We discussed that these results have interesting implications. In fact we have an algorithm which uses these properties to construct the skeleton and to identify the "disconnections"; adds extra edges to the skeleton to make it complete, and uses it to find a feasible path for M from a given initial point to a given final point. This will be described in a later paper.

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