Finding Hamiltonian Circuits in Arrangements of Jordan Curves is NP-Complete

Chuzo Iwamoto

Department of Computer Science and Communication Engineering
Kyushu University
Fukuoka, Japan
iwamoto@csce.kyushu-u.ac.jp

Godfried T. Toussaint

School of Computer Science
McGill University
Montreal, Canada
godfried@opus.cs.mcgill.ca

Abstract

Let $A = \{C_1, C_2, \ldots, C_n\}$ be an arrangement of Jordan curves in the plane lying in general position, i.e., every curve properly intersects at least one other curve, no two curves touch each other and no three meet at a common intersection point. The Jordan-curve arrangement graph of A has as its vertices the intersection points of the curves in A, and two vertices are connected by an edge if their corresponding intersection points are adjacent on some curve in A. We further assume A is such that the resulting graph has no multiple edges. Under these conditions it is shown that determining whether Jordan-curve arrangement graphs are Hamiltonian is NP-complete.

1 Introduction

A *Hamiltonian circuit* in a graph is a circuit which passes through every vertex of the graph exactly once. The *Hamiltonian circuit problem* asks whether there exists at least one Hamiltonian circuit in a given graph. There have been at least three approaches taken in the past towards the study of Hamiltonian circuits. In one approach sufficient conditions are sought for which graphs are Hamiltonian. For example, it is known that all 4-connected triangulated graphs [Wh31], 4-connected planar graphs [Tu56], [Ch85] and 1-sail line arrangement graphs [EGT92] are Hamiltonian. Also, the visibility graphs of sets of line segments with the property that the line segments are of unit length whose endpoints have integer coordinates, are Hamiltonian [OR91]. A related computational question concerns how fast we can find a Hamiltonian circuit in a Hamiltonian graph. For any 4-connected planar graph G with n vertices, a Hamiltonian circuit in G can be found in $O(n^3)$ time [Go82]. If only the vertices where a turn is made need be reported (a *streamlined* Hamiltonian circuit) then a Hamiltonian circuit for 1-sail line arrangement graphs can be found in $\Theta(n \log n)$ time, where n is the number of lines in the arrangement [EGT93]. A second approach is to find restricted classes of graphs for which we can determine in polynomial time whether or not instances of such graphs admit a Hamiltonian circuit. For example if each line segment of a set of n disjoint line segments in the plane has at least one of its end points on the convex hull of the set, it can be determined in $O(n \log n)$ time whether the set admits a Hamiltonian circuit through its endpoints such that it is a simple polygon and uses every line segment exactly once [RIT90]. The third approach to the Hamiltonian circuit problem has been to search for restricted classes of graphs for which the problem is NP-complete. For example, the Hamiltonian circuit problems for general graphs [Ka72], for 3-regular 3-connected planar graphs [GJT76], and for 3-regular bipartite planar graphs [ANS80] are known to be NP-complete. Also, in the line segment problem discussed above, if the convex hull restriction is removed and line segments are allowed to touch at their end points it is NP-complete to determine if they admit a simple Hamiltonian circuit [Ra87]. One of the results on a related problem is the NP-completeness of the edge Hamiltonian path problem for bipartite graphs [LW93].

Recently several different classes of arrangement graphs have been introduced [EGT92]. For example, an arrangement of n lines in general position (no two parallel and no three concurrent) defines a set of intersection points joined by edges. The graph whose vertices are the intersection points and whose edges are the segments of the lines between adjacent intersection points, is called a *line-arrangement graph.* Hazel Everett [EGT93] has shown that not all line-arrangement graphs are Hamiltonian. On the other hand the great-circle-arrangement graph on the sphere (obtained in a similar manner from a set of great circles on the sphere in general position) has been recently shown to be Hamiltonian by Bruce Reed [Re]. In this note we establish the NP-completeness of the Hamiltonian circuit problem for a new class of graphs we call *Jordan-curve arrangement graphs* with no multi-edges. This class is properly contained in the class of 4-regular graphs. Therefore our result is strictly stronger than the NP-completeness result for 4-regular planar graphs.

2 Definitions and Results

Let C_i and C_j denote two Jordan curves. Let $A = \{C_1, C_2, \ldots, C_n\}$ be an *arrangement of Jordan curves.* The set of intersection points of C_i and C_j is denoted by $I(C_i, C_j)$, and the set of intersection points has measure zero. In this paper, we assume that (i) no two curves touch each other (thus, $|I(C_i, C_j)| > 1$ if C_i properly intersects C_j) and (ii) no three curves are concurrent, i.e., share a common intersection point. Let $V(A) = \{v | v \in I(C_i, C_j) \text{ such that } C_i, C_j \in A\}$. The *Jordan-curve arrangement graph* of an arrangement A is the graph $G_a = (V_a, E_a)$ such that (i) $V_a = V(A)$ and (ii) edges in E_a are formed by curves of A (see Fig. 1). Note that a Jordan-curve arrangement graph may contain multi-edges.

Theorem 1. *The Hamiltonian circuit problem for Jordan-curve arrangement graphs with no multi-edges is NP-complete.*

Remark. Since every Jordan-curve arrangement graph is a 4-regular planar graph, the Hamiltonian circuit problem for 4-regular planar graphs is also NP-complete. The class of Jordan-curve arrangement graphs with no multi-edges is properly contained in the class of 4-regular planar graphs, i.e., there exist 4-regular planar graphs G with no multi-edges such that G cannot be formed by any arrangement of Jordan curves (see Fig. 2).

Proof of Theorem 1. Since the Hamiltonian circuit problem for general graphs is in NP [Ka72], the problem for Jordan-curve arrangement graphs with no multi-edges is also in NP. It is known that the Hamiltonian circuit problem for 3-regular planar graphs with no multi-edges is NP-complete [GJT76]. We reduce each 3-regular planar graph G with no multi-edges to a Jordan-curve arrangement graph G_a with no multi-edges such that G is Hamiltonian if and only if G_a is Hamiltonian. The overview of the proof is as follows. Starting with G, (i) we construct 4-regular planar graph

G_1 with *multi*-edges and (ii) we then construct 4-regular planar graph G_a with *no* multi-edges. (iii) We prove that G is Hamiltonian if and only if G_a is Hamiltonian (Lemma 1), and (iv) we then prove that G_a is a Jordan-curve arrangement graph (Lemma 2).

Construction of G_1: We first replace each edge of G by a pair of multi-edges (see Fig. 3). We then replace each vertex v by three vertices (say v_x, v_y, and v_z) and three edges, (v_x, v_y), (v_y, v_z), and (v_z, v_x). We call the subgraph induced by these three edges a *triangle*. We denote the resulting graph by $G_1 = (V_1, E_1)$. By construction, G_1 is a 4-regular planar graph with *multi*-edges. Furthermore, G_1 can be constructed from G in polynomial time.

Construction of G_a: The basic idea is to add four vertices and four edges to each pair of multi-edges (see Figs. 4 and 5). We first find a vertex subset $S \subseteq V_1$ such that (i) exactly one vertex of each pair of multi-edges is in S and (ii) at least one vertex of each triangle is in S. (We will show how to find such an S later.) Suppose that a vertex a of G_1 is in S (see Fig. 4-(a)). Let e_1, e_2, e_3, and e_4 be the edges incident to a in clockwise order. For $1 \leq i \leq 4$, we "divide" edge e_i into two edges by adding a new vertex a_i on e_i (see Fig. 4-(b)). We then add four edges (a_1, a_2), (a_2, a_3), (a_3, a_4), and (a_4, a_1). We call a subgraph induced by these four edges a *circle*. By applying the above procedure to each vertex in S, we obtain G_a (see Fig. 5-(b)). Now G_a has no multi-edges.

It remains to show how to construct $S \subseteq V_1$ in polynomial time. We first construct a vertex set $S_1 \subseteq S$ such that (i) *at most* one vertex of each pair of multi-edges is in S_1 and (ii) *exactly* one vertex of each triangle is in S_1. S can be obtained by finding pairs of multi-edges such that both vertices of each of the pairs are not in S_1 and by adding one arbitrary vertex of each such pair to S_1. The construction of S_1 is as follows. We construct a *directed* subgraph $D = (V, E_D)$ in the original graph $G = (V, E)$ such that all of D's vertices have out-degree one (see Fig. 5-(a)). Recall that vertices and edges in G were replaced by triangles and pairs of multi-edges in G_1, respectively (see Fig. 3-(b)). Each vertex v_x of G_1 is in S_1 if and only if there exists a directed edge (v, x) in E_D (see Figs. 3-(b) and 5-(a)). $D = (V, E_D)$ can be constructed as follows. We first find an undirected spanning tree, say $T = (V, E_T)$, in G. We choose an arbitrary vertex, say r, among T's leaves. We regard r as the new root of T. We then construct a directed spanning tree rooted at r by adding direction information to T. Note that every vertex of T, except for root r, now has outdegree exactly one. (r has in-degree one and out-degree zero.) Furthermore, we find an undirected edge $(r, x) \in E$ such that $(x, r) \notin E_T$. By adding *directed* edge (r, x) to E_T, we obtain E_D (and hence we obtain $D = (V, E_D)$).

Lemma 1. *G is Hamiltonian if and only if G_a is Hamiltonian.*

Proof. ($\Leftarrow$) Let v be a vertex in the 3-regular planar graph G (See Fig. 3-(a)). By the above reduction, vertex v is reduced into a subgraph, say S_v, composed of one triangle and at least one circle (see Fig. 6). It should be noted that removing three vertices, v_1, v_2, and v_3 (shown in Fig. 6), from G_a decomposes G_a into at least two connected components, one of which corresponds to v in G. (Intuitively, v_1, v_2, and v_3 in Fig. 6 correspond to the three edges (v, x), (v, y), and (v, z) in Fig. 3-(a), respectively.) If there is a Hamiltonian circuit in G_a which passes through vertices in S_v from v_1 to v_2, then we can construct the corresponding Hamiltonian circuit in G which passes through v from (x, v) to (v, y). Therefore, if there is a Hamiltonian circuit in G_a, then there is a Hamiltonian circuit in G.

($\Rightarrow$) For each of the possible cases shown in Fig. 6, S_v has a Hamiltonian path between every two of the three vertices, v_1, v_2, and v_3 (see Fig. 7, symmetric cases are omitted). Thus, if there is a Hamiltonian circuit in G, then we can construct the corresponding Hamiltonian circuit in G_a. □

Lemma 2. *G_a is a Jordan-curve arrangement graph.*

Proof. Since G_a was constructed by adding circles to G_1, G_a is a Jordan-curve arrangement graph if G_1 is a Jordan-curve arrangement graph. In the following, we show G_1 is a Jordan-curve arrangement graph. Consider the following edge-coloring algorithm:

(1) Initially, all edges have no colors.
(2) Choose an arbitrary edge with no color, and color it with a new color.
(3) Find an edge, say e_1, with no color which satisfies the following condition: There exist three edges e_2, e_3 and e_4 such that e_1, e_2, e_3, and e_4 are incident to a vertex in clockwise order and that e_3 has already been colored. (See Fig. 4-(b).)
(4) Color e_1 with the same color as e_3.
(5) Repeat (3) and (4) until there is no edge e_1 satisfying the above condition.
(6) Repeat (2)-(5) until all edges are colored.

By applying this algorithm to G_1, we obtain subgraphs each of which consists of edges having the same color. We now show that these subgraphs are 2-regular graphs.

Consider an arbitrary face of G (see Fig. 8-(a)). By the transformation from G to G_1, (i) each vertex of G, which is the boundary point of three faces, is replaced by three edges of a triangle of G_1, and (ii) each edge of G, which is the boundary between two faces, is replaced by a pair of multi-edges (see Fig. 8-(b)). Thus, the edges colored by a single color form a 2-regular planar subgraph which corresponds to a face of G. Hence, G_1 can be formed by an arrangement of Jordan curves. □

Example. We give an example of arrangements of Jordan curves whose graphs are not Hamiltonian (see Fig. 9-(c)). The 3-regular planar graph shown in Fig. 9-(a) is not Hamiltonian, since it is not 1-tough [Ch85], i.e., we can decompose it into *three* connected components by removing *two* vertices. This non-Hamiltonian graph can be reduced to the 4-regular planar graph with no multi-edges shown in Fig. 9-(b). This 4-regular graph is also non-Hamiltonian, since removing two subgraphs which correspond to the above two vertices decomposes the 4-regular graph into three connected components. (Although there is a Hamiltonian path in each of the three connected components, no Hamiltonian *circuit* can be constructed by connecting those three Hamiltonian paths.) Therefore, the arrangement of Jordan curves shown in Fig. 9-(c) is non-Hamiltonian.

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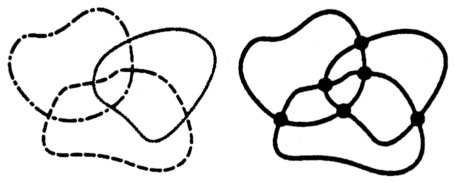


(a) (b)

Fig. 1 (a) Arrangement of three Jordan curves
(b) Jordan-curve arrangement graph

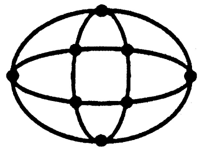
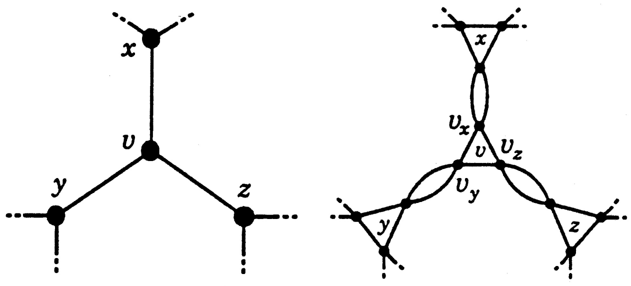
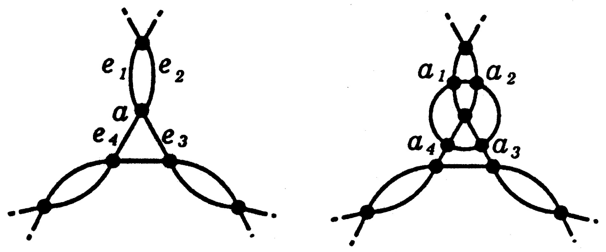


Fig. 2 A graph that cannot be formed by any arrangement of Jordan curves



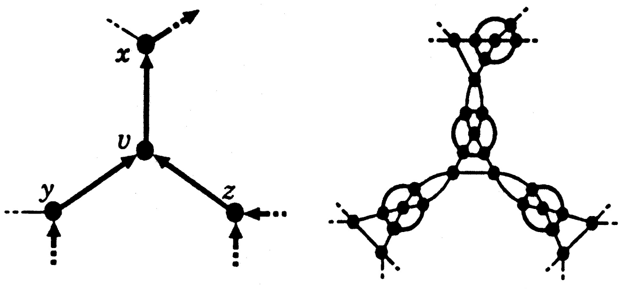
(a) (b)

Fig. 3 (a) Vertices of G (b) Triangles of G_1



(a) (b)

Fig. 4 (a) Four edges incident to vertex a in G_1 (b) Circle in G_a



(a) (b)

Fig. 5 (a) Directed subgraph D in G
(b) Circles of G_a

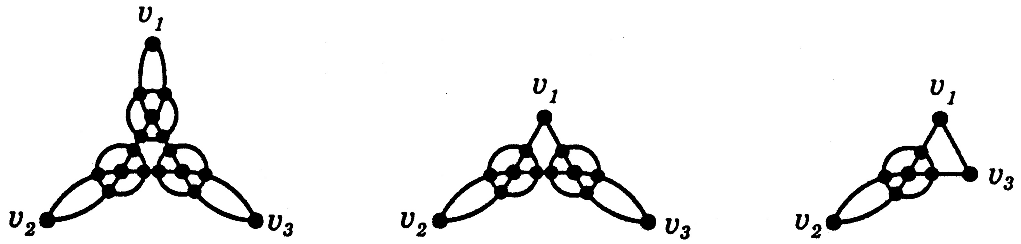


Fig. 6 Three possible cases

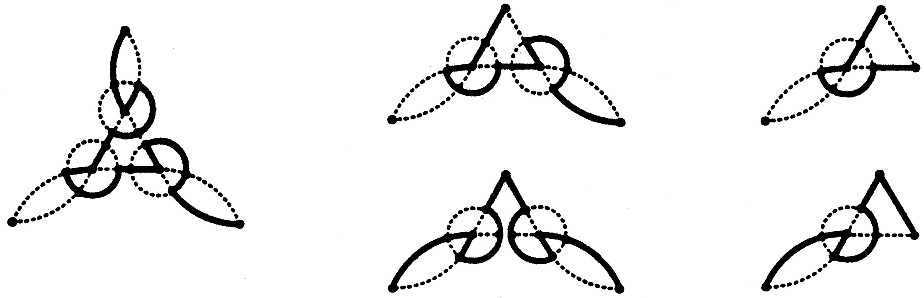


Fig. 7 Hamiltonian paths

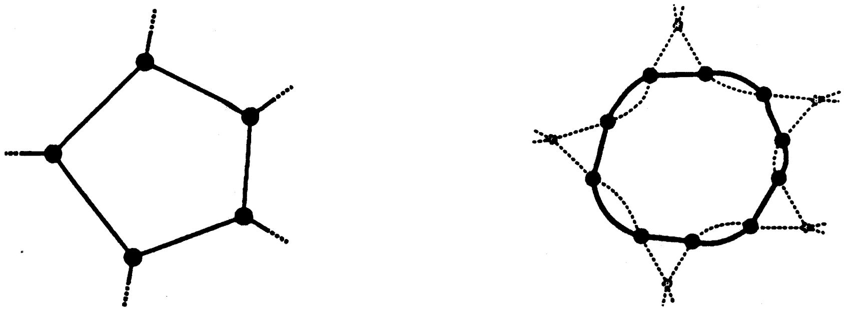
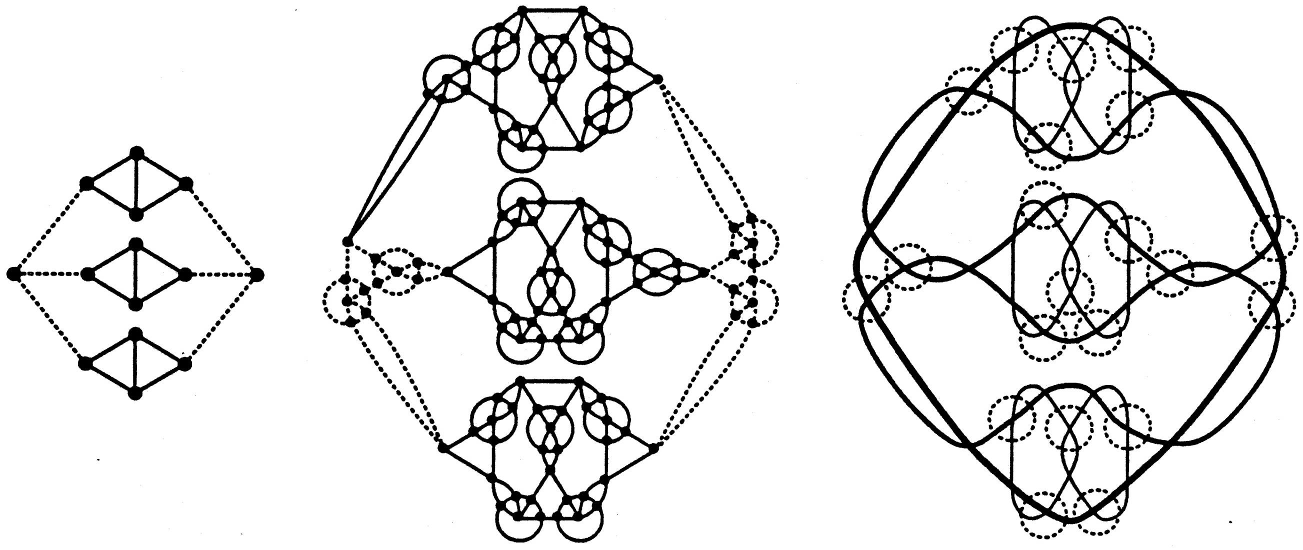


Fig. 8 (a) Face of G (b) Corresponding 2-regular subgraph in G_1



(a) 3-regular graph (b) 4-regular graph with no multi-edges (c) Arrangement of Jordan curves

Fig. 9 Non-Hamiltonian planar graphs and a non-Hamiltonian arrangement