Optimal Floodlight Illumination of Orthogonal Art Galleries*

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Abstract

We show a tight bound of $\lfloor 3(n-1)/8 \rfloor$ for the number of orthogonal floodlights sufficient to cover an orthogonal polygon with n vertices. Our results lead directly to a very simple linear algorithm that computes a covering of the polygon and that guarantees a total aperture of no more than $3\pi n/16$. This is an improvement of a factor of 2 over the naive use of the original theorem for orthogonal art galleries and avoids complex algorithms for partitioning of orthogonal polygons. Moreover, if floodlights can be placed in the boundary of the polygon (not only at vertices), we can reduce this to a tight bound of $\lfloor n/4 \rfloor$ and compute their positions also in linear time.

1 Introduction

The question of guarding a polygonal art gallery has raised many problems ranging from polygon decomposition and problem complexity to combinatorial structure of visibility graphs [6]. Moreover, the study of visibility in this type of geometric setting has not only been naturally motivated by many applications, but it has also been fundamental in developing many theoretical and practical results [8]. Despite the many variants of the problem, little regard has been placed to the assumption that guards can cover a complete 2π range of orientations around them. Rawlins [7] studied visibility along finitely oriented staircases and provided corresponding Art Gallery Theorems. Estivill-Castro and Raman [4] studied edge mobile guards with finite oriented visibility rays. However, only recently the question of studying visibility covers with *floodlights* (that is, static guards with restricted angle of vision) has been raised for covering a line [3] and covering the plane [1].

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In this paper we address the problem of covering the interior of an orthogonal polygon with floodlights. Section 2 describes the floodlight covering problem for art galleries. Section 3 proves that $\lfloor 3(n-1)/8 \rfloor$ orthogonal floodlights are always sufficient and sometimes necessary to cover an orthogonal art gallery. Our proof leads to a linear algorithm for finding a covering, that it is not only simple to implement, but results in a covering with half the total aperture of the naive use of the original art gallery theorem. Section 4 demonstrates that we can reduce the number of floodlights to $\lfloor n/4 \rfloor$ if they can be placed at points in the boundary, and not only at vertices. Section 5 illustrates the difficulties that make the study of floodlight illumination of general polygons a hard problem. Section 6 provides some final remarks.

2 The floodlight covering problem

Consider a polygonal art gallery given by a simple orthogonal polygon P in the plane with n vertices and no holes. Given a set of k floodlights where each has an aperture of $\alpha \in (0, 2\pi]$, the problem consists of determining if it is possible to place the floodlights in k distinct vertices of the polygon and illuminate (cover) its interior. Note that no more than one floodlight is allowed at a vertex of the polygon, otherwise we have one floodlight with larger aperture. In particular, the original version of the orthogonal art gallery theorem establishes that, if $\alpha = 2\pi$, then $\lfloor n/4 \rfloor$ floodlights are always sufficient and sometimes necessary to cover an orthogonal art gallery [5].

The question we address here is what is the corresponding version of the this result for $\alpha \in (0, 2\pi)$.

3 Orthogonal floodlights

Clearly, since in an orthogonal art gallery the angles at the vertices are in $\{\pi/2, 3\pi/2\}$, we immediately have that, if $\alpha \in [3\pi/2, 2\pi]$, then $\lfloor n/4 \rfloor$ floodlights are

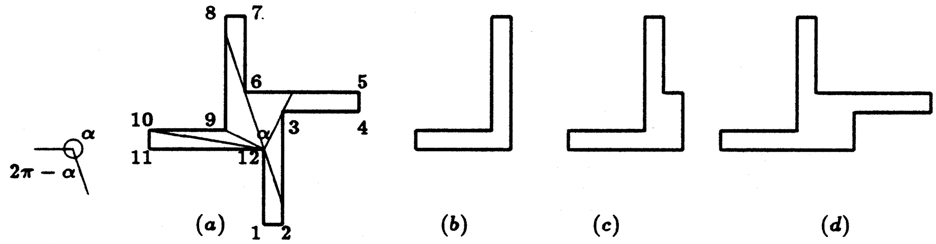

Figure 1: (a) An orthogonal polygon that requires one α-floodlight for each tong, when $\alpha \in [\pi/2, 3\pi/2)$. (b), (c) and (d) Subpolygons that require one, two and three floodlights, respectively.

always sufficient and some times necessary to cover an orthogonal art gallery.

In this section, we study the case when $\alpha \in [\pi/2, 3\pi/2)$. We first establish the necessity of $\lfloor 3(n-1)/8 \rfloor$ floodlights. Let $\alpha \in [\pi/2, 3\pi/2)$. Consider the orthogonal polygon in Figure 1 (a), where the tongs are long enough that a floodlight as shown in the picture can illuminate at most one tong. Clearly, at least 4 floodlights are required to cover this polygon. Moreover, the subpolygon of Figure 1 (b) has one reflex vertex and requires one floodlight. The subpolygon of Figure 1 (c) has two reflex vertices and requires 2 floodlights. The subpolygon of Figure 1 (d) has three reflex vertices and requires 3 floodlights. Now, consider the progression illustrated by Figure 2. We start with a rectangle in Figure 2 (a), which is illuminated with one floodlight. At each new stage, we merge a copy of the polygon in Figure 1 (a) by its left tong. Vertices 10 and 11 are identified with the two right most vertices in the previous figure. It is not hard to see that at each stage 3 more α-floodlights are needed but only 8 vertices are added. Moreover, since $2r + 4 = n$ [6], where r is the number of reflex vertices, the last polygon in the sequence can be one of the subpolygons in Figure 1 (b)-(d). It now follows that, for all r, $\lfloor (6r+1)/8 \rfloor + 1$ floodlights are necessary. That $\lfloor 3(n-1)/8 \rfloor$ floodlights are necessary for all n now follows from arithmetic manipulation.

We now prove that $\lfloor 3(n-1)/8 \rfloor$ floodlights, with $\alpha = \pi/2$ are always sufficient to illuminate an orthogonal art gallery with n vertices. We use the following notation introduced by Rawlins [7]. Given an orthogonal polygon P, an edge e of P is said to be a *North edge* (N-edge for short), if the interior of the polygon is immediately below e. East, West and South edges are defined analogously. A vertex is said to be a *North-East vertex* (NE-vertex for short), if the

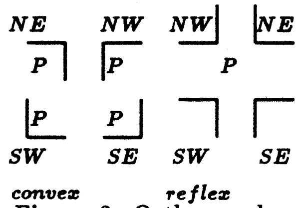

Figure 3: Orthogonal polygons may have these 8 types of vertices.

polygonal edges that intersect at the vertex are an N-edge and an E-edge. NW-vertices, SE-vertices and SW-vertices are defined similarly. Figure 3 displays the eight possible types of vertices in an orthogonal polygon.

We define the following floodlight placement rule.

> *North-East placement rule* (NE-rule): For each North edge e of the polygon, place a floodlight aligned with e at the East vertex of e. For each East edge e of the polygon, place a floodlight aligned with e at the North vertex of e.

For a diagram of the NE-rule see Figure 4 (a).

Proposition 3.1 *The NE-rule produces an assignment of floodlights that illuminates the interior of P.*

Proof: Let p be a point in the interior of the polygon P. Let x be the first point in the border of P visible by a horizontal ray from p to the East. Clearly, x is in an E-edge e and p is visible from x. Now, consider a point x' in e just above x and consider the rectangle R with extreme points at x' and p; see Figure 4 (b). Clearly, if x' is close enough to x, the rectangle R is contained in P. Consider moving x' North until it cannot be moved further without R leaving the interior of P. This happens because

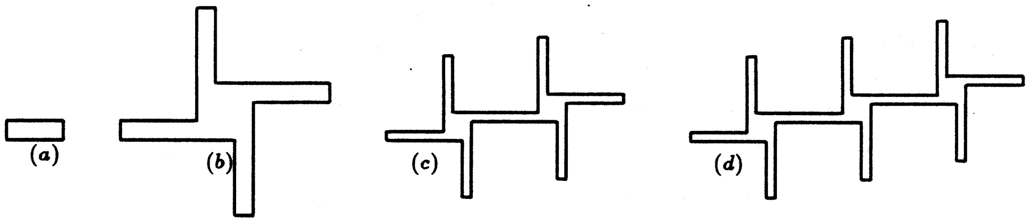

Figure 2: Orthogonal polygons that require one α-floodlight for each tong, when $\alpha \in [\pi/2, 3\pi/2)$.

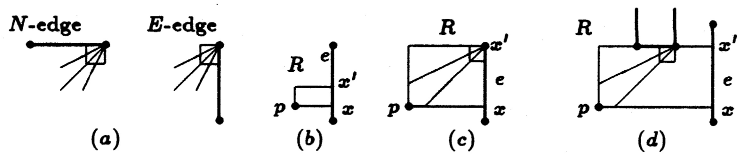

Figure 4: Diagram illustrating the placement of floodlights by the NE-rule

- x' has reached the North vertex of e, in which case, p is illuminated by a floodlight at this point; see Figure 4 (c), or
- the upper side of the rectangle R has coincided with a North edge, in which case, p is illuminated by a floodlight at the East point of this North edge; see Figure 4 (d).

In both cases, p is illuminated and the proof is complete. □

Similarly, we can define a NW-rule, a SE-rule and a SW-rule, each illuminating the polygon. We are now ready to prove sufficiency.

Proposition 3.2 *If $\alpha = \pi/2$, then $\lfloor 3(n-1)/8 \rfloor$ floodlights are sufficient to illuminate an orthogonal polygon P.*

Proof: Illuminate the polygon by each of the four rules proposed above. Let $||X||$ denote the number of floodlights used by the X rule. Note that each edge of the polygon receives at most two floodlights (for example, a N-edge receives a floodlight at its E-vertex in the NE-rule and at its W-vertex in the NW-rule). Moreover, in the NE-rule, a NE-convex vertex receives only one floodlight. Thus, the number $||NE||$ of floodlights used by the NE-rule is given by

$$||NE|| = ||N||_e + ||E||_e - ||NE||_c,$$

where $||N||_e$ is the number of N-edges, $||E||_e$ is the number of E-edges and $||NE||_c$ is the number of NE-convex vertices. Thus, the total number of floodlights used by the four rules is given by

$$||NE|| + ||NW|| + ||SE|| + ||SW|| = 2n - c,$$

where c is the number of convex vertices in the polygon. Since $c = (n+4)/2$ [6], we have that $||NE|| + ||NW|| + ||SE|| + ||SW|| = (3n-4)/2$. Now, note that each of the rules defines a disjoint set of floodlights; thus, there is one of the four rules that uses no more that $\lfloor 3(n-1)/8 \rfloor$ floodlights. Placing floodlights according to the rule with least floodlights gives the result. □

We have proved the following result.

Theorem 3.3 *Let P be an orthogonal polygon with n vertices and $\alpha \in [\pi/2, 3\pi/2)$, then $\lfloor 3(n-1)/8 \rfloor$ floodlights are always sufficient and some times necessary to illuminate P.*

Theorem 3.3 proves that a total aperture of $3\pi n/16$ is always sufficient to illuminate (guard) an orthogonal polygon. We claim that this result is of significant relevance despite the fact that it may suggest that more guards are required than in the original art gallery theorem. We support this claim with three observations:

1. Using Theorem 3.3, the total aperture of $3\pi n/8$ naively proposed by the original version of the orthogonal art gallery theorem has been reduced by half.
2. The placing rules lead directly to a linear algorithm that is much simpler than the algorithms for guards that require sophisticated trapezoidization, quadrilateralization or decomposition into L-shaped pieces [6].

3. The placing rules does not always place $3n/8$ floodlights but it may place much less; for example, in a staircase polygon, only one floodlight is used.

4 Illuminating with $\lfloor n/4 \rfloor$ orthogonal floodlights

An orthogonal art gallery P with r reflex vertices can be partitioned into $\lfloor r/2 \lfloor +1$ L-shaped pieces [6], and since each L-shaped piece can be illuminated with one floodlight, we have that $\lfloor n/4 \rfloor$ orthogonal floodlights are sometimes necessary and always sufficient to illuminate an orthogonal polygon. This seems to contradict our previous results; however, the missing details is that using O'Rourke's algorithm to partition P into L-shaped pieces, some of the floodlights will be placed in the interior of the polygon. This seems rather unsatisfactory.

In this section, we show that we can illuminate the polygon P with $\lfloor n/4 \rfloor$ orthogonal floodlights placed at points in the boundary of P. We prove this by showing that $\lfloor r/2 \rfloor + 1$ orthogonal floodlights at points in the boundary are always sufficient to illuminate P, where r is the number of reflex vertices in P.

Necessity is established by the well-known "comb" example [6, Figure 2.18]. Sufficiency follows an inductive argument similar to O'Rourke's proof of the orthogonal art gallery theorem [6, Sections 2.5 and 2.6].

A *horizontal cut* of an orthogonal polygon P is an extension of the horizontal edge incident to a reflex vertex through the interior of the polygon. A cut *resolves* a reflex vertex in the sense that the vertex is no longer reflex in either of the two pieces of the partition determined by the cut. Clearly, a cut does not introduce any reflex vertices. A horizontal cut is an *odd-cut* (also and H-odd-cut) if one of the halves contains an odd number of reflex vertices.

Proposition 4.1 *Let P be an orthogonal polygon and partition P into $P_1, P_2, \ldots, P_t$ by drawing all H-odd-cuts and all H-cuts that are visibility rays of two reflex vertices. Then, each P_i is in general position and can be covered with $\lfloor r_i/2 \rfloor + 1$ floodlights in the boundary of P, where r_i is the number of reflex vertices in P_i.*

Each P_i, for $i = 1, \ldots, t$ is a polygon with no H-odd-cuts, and thus, is formed by two histograms joined at their bases [6]. Moreover, the structure of

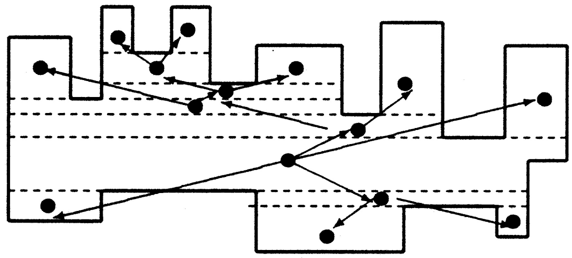

Figure 5: The H-tree of a polygon in general position with no H-odd-cuts.

the polygons P_i, for $i = 1, \ldots, t$, can be determined further in term of their H-graph. We will profit of the tree structure of this H-graph to prove Proposition 4.1.

Call a reflex vertex *H-isolated* if the other endpoint of its incident horizontal edge is not reflex, and otherwise call it an *H-pair*. The H-graph of P_i is constructed as follows. For each H-pair introduce H-cuts for both reflex vertices in the pair. The H-graph is the adjacency graph of the partition defined by the H-cuts at H-pairs. Each piece of the partition corresponds to a node in the graph, and a node A is connected by an arc directed to a node B iff (1) A and B are adjacent pieces separated by an H-cut, and (2) the H-pair corresponding to the H-cut lies on the boundary of the A piece.

The H-graph of a P_i is a tree, but since P_i has no H-odd-cuts, then P_i has exactly one H-isolated vertex, located at a region that is the only region corresponding to a source node in the H-graph [6]; see Figure 5.

To prove Proposition 4.1 we first note that all vertical edges of P_i are in the boundary of P, for $i = 1, \ldots, t$. We will prove Proposition 4.1 by introducing a transformation T that

1. replaces two leaves and a branch of the H-tree for a leaf,
2. places an orthogonal floodlight at a reflex vertex called d in a region that is not a leaf,
3. resolves vertex d and introduces a reflex vertex d',
4. removes two other reflex vertices called a and b, and
5. produces a new polygon with two less reflex vertices, no H-odd-cuts and such that all the vertical

edges of regions in the H-tree that are not leaves are edges of the original polygon.

Figure 6 illustrates the transformation.

An inductive application of transformation T proves the result since condition 4 guarantees the bound of $\lfloor r_i/2 \rfloor + 1$ floodlights to illuminate P_i, and conditions 2 and 5 guarantee that floodlights are always placed in points of the boundary of the original polygon. Condition 3 guarantees that no two floodlights are placed at the same point.

Since the H-graph can be constructed in linear time [6] and trapezoidization can be achieved in linear time [2], it is not hard to see that the above argument results in a linear algorithm. Thus, we have proved the following result.

Theorem 4.2 *Let P be an orthogonal polygon with n vertices, then $\lfloor n/4 \rfloor$ orthogonal floodlights placed in the boundary of P are always sufficient and sometimes necessary to illuminate the interior of P. Moreover, such set of floodlights can be found in $O(n)$ time.*

5 The difficulty of the general case

Since each triangle has an angle of aperture at least $\pi/3$, it seems that by placing a $\pi/3$ floodlight at each of the corresponding triangles of a triangulated polygon a cover of the polygon will be obtained. Although this argument demonstrates that $n\pi/3$ is enough total aperture, it does not provide a floodlight illumination, since two floodlights may be required at the same vertex.

In this section, we attempt to illustrate that the solution to the floodlight illumination problem of a polygon requires the study of different partitioning strategies than those considered so far, like triangulation. We will construct a polygon that requires that some α-floodlights be placed not aligned with any of the polygonal edges. We will prove the following result.

Proposition 5.1 *For $\alpha \in [0, \pi)$, there are polygons P that cannot be illuminated by α-floodlights, unless some of the floodlights are not aligned with the polygonal edges.*

Proof: Consider $\alpha < 3\pi/4$. Then, there is $\epsilon > 0$ such that $\beta = \alpha + \epsilon < e\pi/4$. Imagine a circle C of radius r and centered at $(0,0)$ in the Cartesian plane. This circle will be inside the polygon.

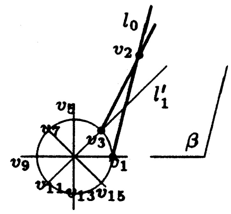

Figure 7: The construction of a polygon that requires floodlights not aligned with its edges.

The first vertex v_1 of the polygon is at $(r, 0)$; the vertex v_3 is on the circle C and the ray l'_1 that makes an angle of $\pi/4$ with the x-axis. For $i = 0, 1, \ldots, 7$, vertex v_{2i+1} is on the intersection of C and the ray l'_i that makes an angle of $i\pi/4$ with the x-axis.

The edge (v_1, v_2) lies on the line l_0 that forms an angle of $\pi - \beta$ with the x-axis and passes through v_1. Since $\pi - \beta = \pi - (\alpha + \epsilon) > \pi - 3\pi/4 = \pi/4$, the line l_0 intersects with the ray l'_1. The vertex v_2 is on the opposite half-plane of the line l'_1 than v_1; see Figure 7. The vertex v_{2i} is defined analogously, rotating clockwise by $\pi/4$ the construction in Figure 7.

We claim that any α-floodlight aligned with any of its edges does not illuminate the center of C. Because of symmetry, we just need to analyze what an α-floodlight at v_2, at v_1 aligned with the edge v_1, v_2 and at v_3 aligned with the edge v_3, v_2 can illuminate.

Clearly, an α-floodlight at v_2 cannot illuminate $(0,0)$ because v_3 is on l'_1 and v_2 is on the opposite side. An α-floodlight at v_1 aligned with the edge v_1, v_2 will not illuminate $(0,0)$ since the angle at v_1 is $\beta > \alpha$. Finally, an α-floodlight at v_3 aligned with v_3, v_2 cannot illuminate $(0,0)$ since the angle $v_2 v_3 (0,0)$ at v_3 is larger than π and $\alpha < 3\pi/4$.

Adding more tongs to the polygon, the example can be extended for $\alpha < \pi$. □

6 Final Remarks

We have shown a tight bound of $\lfloor 3(n-1)/8 \rfloor$ for the number of orthogonal floodlights sufficient to cover an orthogonal polygon with n vertices. Our results have lead directly to a very simple linear algorithm that computes a covering of the polygon and that guarantees a total aperture of no more than $3\pi n/16$.

However, several open problems remain.

1. What bounds can be found for other classes of polygons?

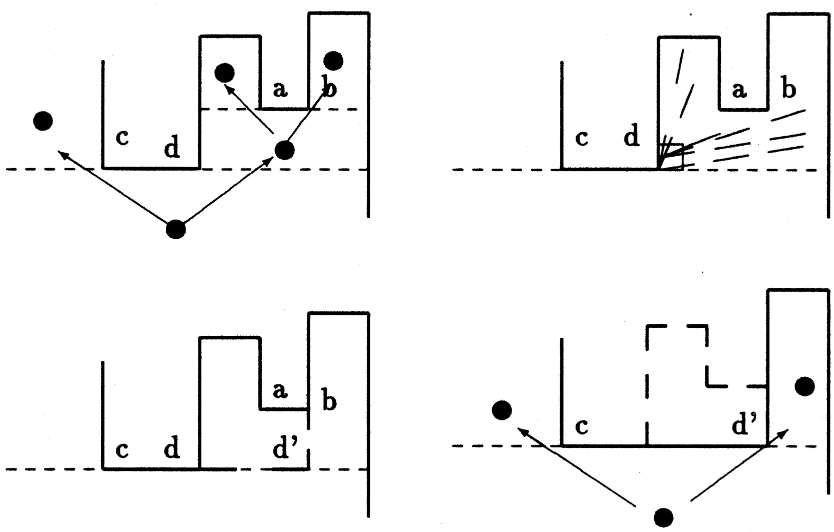

Figure 6: The transformation T that eliminates two reflex vertices, and replaces two leaves and a branch for a region that is a leaf.

2. Is computing the minimum set of covering α-floodlights an NP-hard or NP-complete problem?

3. If the floodlights are allowed to have each a different aperture α_k, what can be said about the problem of finding a cover that optimizes the total angle power given by $\sum_{i=1}^{k} \alpha_k$?

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