Planar Visibility Graphs*

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ABSTRACT

The recognition problem for visibility graphs is, given a graph, to determine whether this graph is the visibility graph of a simple polygon. It is not known whether this problem could be solved in polynomial time. It is even not known whether the recognition problem for *planar* visibility graphs could be solved in polynomial time. In this paper, we shall give the necessary and sufficient conditions for a polygon to have a planar visibility graph. Using this result, we can prove that no 4–connected planar graph could be a visibility graph of a simple polygon.

INTRODUCTION AND TERMINOLOGY

All polygons discussed in this paper are assumed simple. Two vertices of a polygon P are said *visible* from each other (or they can *see* each other) if the closed line segment joining them does not intersect the exterior of P. (Note that the line segment may touch the boundary of P several times.) The *visibility graph* of a polygon P, denoted by G_p, is the graph whose vertices correspond to the vertices of P and two vertices of G_p are joined by an edge if and only if their corresponding vertices in P are visible.

O'Rourke [2] believes that "some of the fundamental unsolved problems involving visibility in computational geometry will not be solved until the combinatorial structure of visibility is more fully understood. Perhaps the purest condensation of this structure is a visibility graph."

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A necessary condition for a graph to be a visibility graph is the existence of at least one Hamiltonian cycle, corresponding to the boundary of the polygon. In [4], Tutte proved that every 4–connected planar graph has a Hamiltonian cycle. In this paper, we shall give the necessary and sufficient conditions for a polygon to have a planar visibility graph. Using this result, we can also prove that no 4–connected planar graph could be a visibility graph; this property has also been derived by Abello, Lin and Pisupati [1]. For literature of the visibility problem, please refer to [3].

The following terminologies will be used throughout this paper. Let P be a polygon. Two vertices U and V of P will divide the boundary of P into two chains: one is called *UV–chain* and the other, *VU–chain*. UV–chain contains those vertices between U and V (with U and V being *excluded*) when the boundary of P is traversed clockwise from U to V. VU–chain is defined similarly. We shall use ΔABC to denote a triangle formed by three vertices A, B, C of P. A triangle ΔABC is called an *empty triangle* if A, B, C can see each other and no other vertex of P lies entirely within ΔABC (it may lie on the boundary of ΔABC).

The three vertices of an empty triangle ΔABC will divide the boundary of a polygon P into three chains: AB–chain, BC–chain, and CA–chain. For convenience, in the following discussion when we say "the three chains of ΔABC," we mean AB–chain, BC–chain, and CA–chain. When we say "AB–chain can see BC–chain," we mean there is a vertex on AB–chain which can see a vertex on BC–chain. "BC–chain can see CA–chain" and "CA–chain can see AB–chain" are defined similarly.

<u>PROPERTIES OF A POLYGON WHICH HAS A PLANAR VISIBILITY GRAPH</u>

Let P be a polygon and let ΔABC be an empty triangle of P. If G_p is planar, we can prove that P satisfies the following properties.

Property 1. Each of the three chains of ΔABC has <u>at most one</u> vertex that can see all of A, B, C. (Otherwise, G_p contains a subdivision of K_5.)

Property 2. P has <u>at most two</u> vertices that can see all of A, B, C. (Otherwise, G_p contains a subdivision of $K_{3,3}$.)

Property 3. If two of the three chains of ΔABC can see each other, then these two chains have exactly one vertex that can see all of A, B, C.

Property 4. If one of the three chains of ΔABC can see both of the other two chains of ΔABC, then P has exactly one vertex V that can see all of A, B, C, and V lies on the chain which can see both of the other two chains.

Property 5. It is impossible for all of the three chains of ΔABC to see each other.

MAIN RESULTS

Let P be a simple polygon which has a *planar* visibility graph and let ΔABC be an empty triangle of P. We shall introduce six possible structures of ΔABC, denoted as structure (0), structure (1a), structure (1b), structure (1c), structure (2a) and structure (2b). See Figure 1 as an illustration.

Structure (0). There is no vertices on the three chains of ΔABC that can see all of A, B, C. (Note that in this case, the three chains can not see each other. Otherwise, from Property 3, there is a vertex can see all of A, B, C.)

Structure (1a). There is only one vertex V on the three chains of ΔABC that can see all of A, B, C, and all the three chains can not see each other. (For example, in Figure 1, P has only one vertex V that can see all of A, B, C, and V is on AB–chain.)

Structure (1b). There is only one vertex V on the three chains of ΔABC that can see all of A, B, C, and only two of the three chains can see each other. (For example, in Figure 1, AB–chain and BC–chain can see each other. From Property 3, V is on AB–chain or BC–chain. In this example, V is on AB–chain.)

Structure (1c). There is only one vertex V on the three chains of ΔABC that can see all of A, B, C, and only one of the three chains can see the other two chains. (For example, in Figure 1, AB–chain can see both BC–chain and CA–chain. From Property 4, V is on AB–chain.)

Structure (2a). There are two vertices U and V on the three chains of ΔABC that can see all of A, B, C, and all the three chains can not see each other. (From Property 1, U

and V lie on different chains. In Figure 1, U is on AB–chain and V is on BC–chain.)

Structure (2b). There are two vertices U and V on the three chains of ΔABC that can see all of A, B, C, and only two of the three chains can see each other. (Note that U can not see V; otherwise, G_p contains a subdivision of K_5.) (For example, in Figure 1, AB–chain and BC–chain can see each other. From Property 3, only one of U and V is on AB–chain and BC–chain. In this example, U is on AB–chain and hence V is on CA–chain.)

Theorem 1. G_p is planar if and only if every empty triangle ΔABC of P has structure (0), structure (1a), structure (1b), structure (1c), structure (2a) and structure (2b).

Proof: *Necessity*: From Property 2, there are only three possible cases:

(i) There is no vertex on the three chains of ΔABC that can see of all of A, B, C. In this case, ΔABC has structure (0).

(ii) There is only one vertex on the three chains of ΔABC that can see all of A, B, C. In this case, if G_p is planar, then from Property 3, Property 4, and Property 5, ΔABC has structure (1a), structure (1b), or structure (1c).

(iii) There are two vertices on the three chains of ΔABC that can see all of A, B, C. In this case, if G_p is planar, then from Property 1, Property 3, Property 4, and Property 5, ΔABC has structure (2a) or structure (2b).

Therefore if G_p is planar, every empty triangle of P has one of the above six structures.

Sufficiency: We can prove by induction on the number of vertices of P that if every empty triangle of P has one of the above six structures, then we can find a planar embedding for G_p. Hence G_p is planar. This part of proof is omitted here. Q.E.D.

Every planar visibility graph must be 2–connected. It is not difficult to construct 3–connected planar visibility graphs. A necessary condition for a graph to be a visibility graph of a polygon is the existence of at least one Hamiltonian cycle, corresponding to the boundary of the polygon. In [4], Tutte Proved that every 4–connected planar graph must have a Hamiltonian cycle. However, we can use Theorem 1 to prove the following theorem.

Theorem 2. No 4–connected planar graph could be a visibility graph.

The proof of Theorem 2 is omitted here. This property was also derived by Abello, et. al. [1] using a different approach. We believe that the six structures proposed in this paper will be useful for completely solving the recognition problem for planar visibility graphs.

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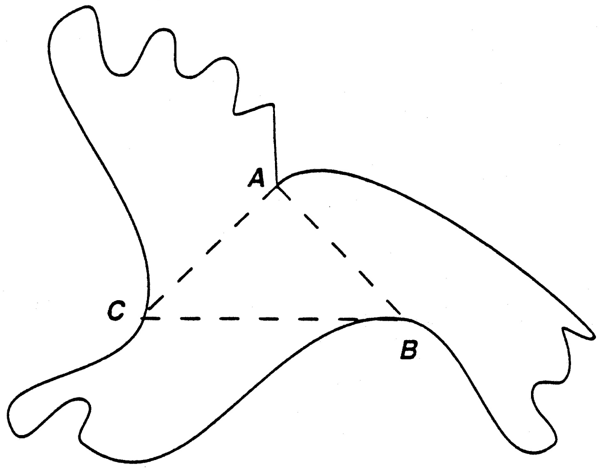
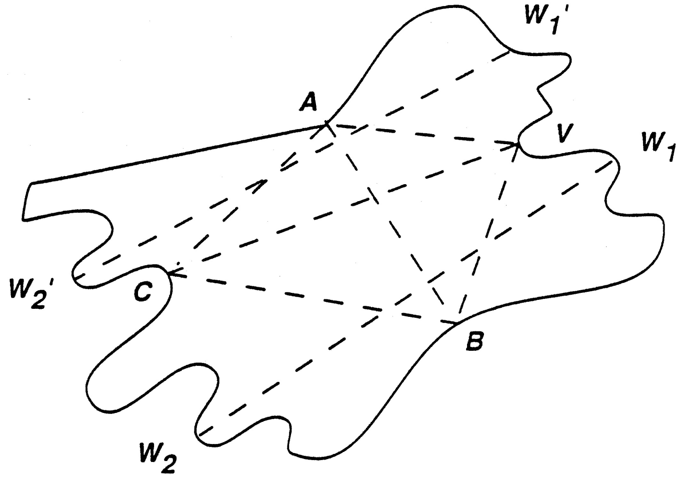
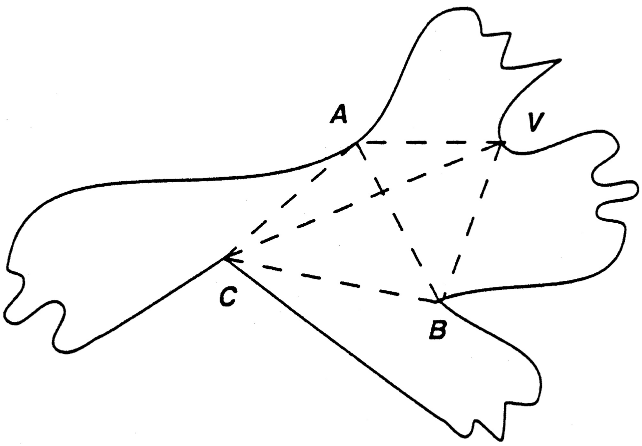


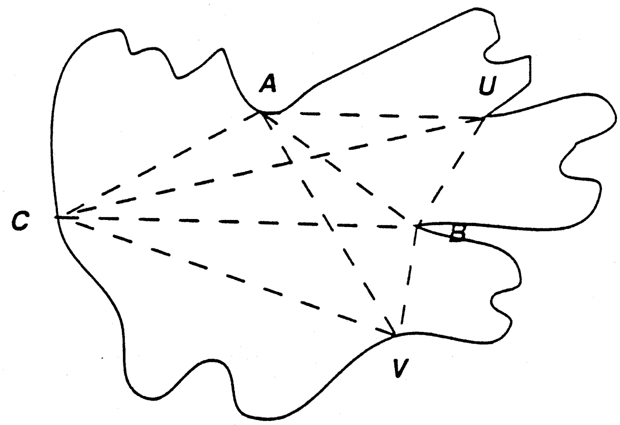
Structure (0)



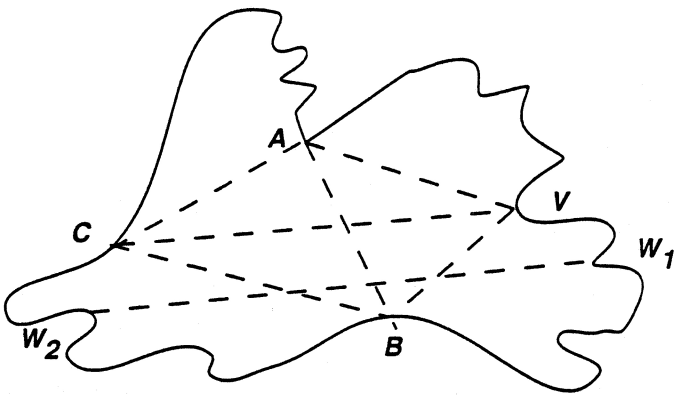
Structure (1c)



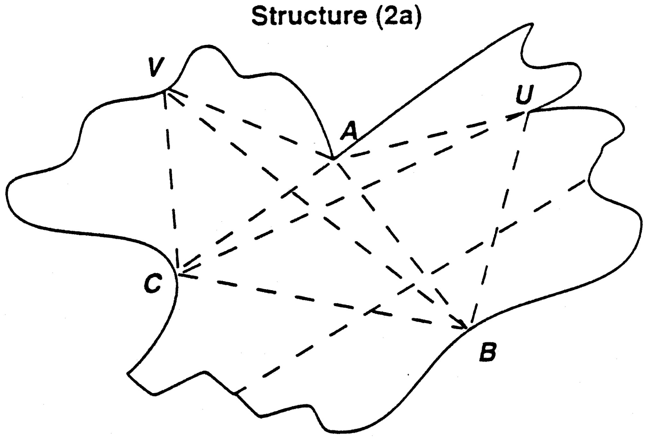
Structure (1a)



Structure (2a)



Structure (1b)



Structure (2b)

Figure 1