Interval Graphs as Visibility Graphs of Simple Polygons Part I: Parachutes

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1 Introduction

Given a simple polygon, its *visibility graph* has the same vertex set as the polygon, with two vertices joined by an edge iff they "see" each other in the polygon—i.e. the line segment joining them stays entirely inside the polygon. The problem of characterizing or recognizing those graphs that are visibility graphs of simple polygons has been open for some time now, see [O87] and [O93]. The problem remains open even when the Hamiltonian cycle that must form the polygon boundary is specified in the graph. Ghosh [G88] gave some necessary conditions for this case.

In this paper we concentrate on the class of interval graphs. One reason for interest in interval graphs is the nice characterization of visibility graphs of spiral polygons as a subset of interval graphs given by Everett and Corneil [EC90]. We define a *parachute* to be a special interval graph with a specified Hamiltonian cycle. Our main result is a characterization and a polynomial-time algorithm to test if a parachute is the visibility graph of a polygon with the specified cycle as its boundary. The class of parachutes is rich enough to require going outside the class of spiral polygons.

A *parachute* is a graph G together with a Hamiltonian cycle $H = \{u, v_1, \dots, v_n\}$ where u is a universal vertex (adjacent to all other vertices) and where, if (v_i, v_j) is an edge with $i < j$, then the set $\{v_i, \dots, v_j\}$ forms a clique in G. The chain $v_1, \dots, v_n$ is called the *parachute chain*.

In order to decide which parachutes are visibility graphs, we will begin in section 2 by characterizing parachutes in terms of interval systems. Essentially, we show that parachutes are interval graphs whose intervals (apart from the universal one) can be ordered so that as we move from one interval to the next, the right and left endpoints never move backward.

More formally, a *clique interval system* is an ordering of the maximal cliques of an interval graph such that for every vertex v the set of maximal cliques to which v belongs occurs consecutively in the ordering. Let $L(v)$ denote the first clique of the ordering containing vertex v, and let $R(v)$ denote the last clique of the ordering containing vertex v. Then v corresponds in the interval system to the interval with left endpoint $L(v)$ and right endpoint $R(v)$.

With respect to a clique interval system, vertex v is *before* w, $v \preceq w$, if $R(v) \leq R(w)$ and $L(v) \leq L(w)$. Note that this is a partial order, not a total order. In particular, intervals i and j with $L(i) < L(j)$ and $R(j) < R(i)$ are not ordered in $\preceq$. In this case we say that i *dominates* j, or that i, j are a *domination pair*. If an interval system has no domination pairs then the ordering $\preceq$ provides a total order (arbitrarily break ties for equal intervals). We call this a *progressive ordering* of intervals. In section 2 we show that parachutes are exactly the graphs with a clique interval system where the intervals (other than the universal one) have a progressive ordering.

Domination pairs seem particularly interesting in characterizing interval visibility graphs. Everett and Corneil, in their characterization of the visibility graphs of spiral polygons as a subclass of interval graphs, showed that a pair i, j with j dominating i can only occur with i in the reflex chain and j in the convex chain. Forbidding all domination pairs gives the class of *proper* interval graphs. In his thesis [C92], Seung-Hak Choi showed that any 2-connected proper interval graph is the visibility graph of a spiral polygon, and he characterized exactly which spiral polygons are obtained this way. Domination pairs

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are permitted in parachutes that are visibility graphs, but only in a restricted way: the universal interval is the only interval that may dominate other intervals (see section 2).

In order to characterize which parachutes are visibility graphs, we will identify a structure called a *knot*, which we will show can only be realized as a convex chain in a polygon. We will show that a parachute is a visibility graph iff it has at most one knot and that knot has a restricted form.

Let $c_1, \ldots, c_m$ be a clique interval system. A *knot* is an interval sub-system consisting of cliques $\mathcal{C}$ and intervals $\mathcal{I}$ where $\mathcal{C}$ is a maximal sequence of maximal cliques, $c_i, \ldots, c_j$ with $j > i$ such that each clique in $\mathcal{C}$ has at least three vertices and any pair of consecutive cliques in $\mathcal{C}$ has at least two vertices in common, and where $\mathcal{I}$ consists of all vertices in all cliques of $\mathcal{C}$.

We will need a few more definitions: A *private* vertex of a graph is a vertex that belongs to exactly one maximal clique. A *link* vertex belongs to exactly two maximal cliques, and a *star* vertex belongs to three or more maximal cliques. (In terms of a clique interval system, private intervals have length one, link intervals length two, and star intervals length at least three.) Two vertices are *duplicate* if they are in precisely the same set of cliques.

A *link-chain* is a path of vertices (no duplicates allowed) where the internal vertices are link vertices and the first and last vertices are either link or private. A *private-link chain* is a path obtained from a link chain by the possible insertion of private vertices (which may be duplicate) between successive vertices.

We will make use of Ghosh's necessary conditions for a graph plus Hamiltonian cycle to be a visibility graph.

Vertex v_j is a *blocking vertex* for non-adjacent pair (v_i, v_k) $i < j < k$, if every vertex along $(v_i \ldots, v_{j-1})$ is non-adjacent to every vertex along $(v_{j+1}, \ldots, v_k)$. Two non-adjacent pairs (v, w) and (v', w') are *separable* with respect to vertex a if v and w are encountered before v' and w' along the Hamiltonian cycle when traversed from a. Ghosh [G88] proved that if graph G with Hamiltonian cycle H is the visibility graph of a polygon with boundary H then they satisfy three necessary conditions. We will need only conditions 2 and 3:

- **Condition 2.** Every non-adjacent pair has a blocking vertex (which must be reflex).
- **Condition 3.** If two non-adjacent pairs have a as their sole blocking vertex, they cannot be separable with respect to a.

2 Characterizing Parachutes in Terms of Clique Interval Systems

In this section we give a characterization of parachutes in terms of clique interval systems. In the remainder of the paper we will use this characterization to decide when a parachute is a visibility graph.

Let graph G be a parachute with Hamiltonian cycle $H = \{u, v_1, \ldots, v_n\}$. By definition, whenever (v_i, v_j) is an edge, $i < j$, the set of vertices $\{v_i, v_{i+1}, \ldots, v_j\}$ forms a clique. Let c be a maximal clique of G. Then c contains u. Not counting u, let the first vertex of c be v_i and the last vertex v_j. Then c must be exactly $\{u, v_i, \ldots, v_j\}$. Observe from this that two maximal cliques cannot have the same first vertex. Thus we can order the maximal cliques of G by their first vertices. It is easy to see that this ordering of cliques provides a clique interval system for G. This will be called the *inherent* clique interval system provided by G and H. (Note that we can efficiently find the inherent clique interval system.) From the ordering $v_1, \ldots, v_n$ specified in H we obtain an ordering of the intervals (except one universal interval). It is easy to see that this ordering is a progressive ordering of intervals.

The above discussion proves:

Theorem 2.1 *Let G be a parachute with Hamiltonian cycle $H = \{u, v_1, \ldots, v_n\}$ where $v_1, \ldots, v_n$ forms a parachute chain. The inherent clique interval system for G and H has a universal interval corresponding to u, and its remaining intervals, corresponding to $v_1, \ldots, v_n$ are progressively ordered. Conversely, any clique interval system whose intervals, apart from one universal interval, can be progressively ordered, has an interval graph that is a parachute.*

Since a progressively ordered set of intervals cannot contain a domination pair, this implies that in the inherent clique interval system of a parachute, the only interval that can dominate another interval is a universal interval.

3 Which Parachutes are Visibility Graphs

In this section we characterize the parachutes that are visibility graphs. Section 3.1 shows that a knot-free parachute is a visibility graph of a simple polygon. Section 3.2 shows that a parachute that is a visibility graph can have at most one knot, and that knot must have a special structure. Section 3.3 shows how to construct a simple polygon out of a parachute with such a knot.

3.1 Knot-Free Parachutes

In this section we show that a knot-free parachute's parachute chain is a private-link chain. Next we show that such a parachute is the visibility graph of a simple polygon.

Lemma 3.1 *In a parachute any star vertex other than the universal vertex u is part of a knot, along with all the cliques the star is contained in.*

PROOF Let c_i be a clique containing a star vertex $s \neq u$, where in the inherent clique ordering system c_i is neither the first nor the last clique containing s. Since the inherent clique interval system has a progressive ordering, c_i does not contain a private vertex. Hence there must exist two vertices v and w such that c_i is the left endpoint of interval v and the right endpoint of interval w. Thus $c_{i-1} \cap c_i \supseteq \{w,s\}$ and $c_i \cap c_{i+1} \supseteq \{v, s\}$. In addition, since all cliques are maximal, some interval must end at c_{i-1}, and some interval must start at c_{i+1}. Thus each of the cliques $\{c_{i-1}, c_i, c_{i+1}\}$ contains at least three vertices, and each consecutive pair has at least two vertices in common. Thus some knot contains s and all the cliques that contain s. □

Lemma 3.2 *In a parachute any duplicate pair of link vertices are part of a knot, along with both the cliques they are contained in.*

PROOF Similar to the above. □

These two lemmas together imply:

Theorem 3.3 *A knot-free parachute's parachute chain is a private-link chain.*

Based on this characterization we can prove that every knot-free parachute is the visibility graph of a simple polygon. We first consider a parachute whose parachute chain is a link chain. Such a parachute is shown to be the visibility graph of an ***umbrella***, of a ***single-mast***, and of a ***twin-mast***.

An ***umbrella*** is a spiral polygon, star-shaped from its sole convex vertex (see Fig. 1). A ***twin-mast*** is depicted in Fig. 2. A ***single-mast*** is formed from a twin-mast by coalescing v_i and v_{i+1}. All three of these polygon types are star-shaped from u and every other vertex sees only its predecessor and successor. Thus in all cases the visibility graph is precisely a parachute whose parachute chain is a link chain.

In the other direction, if we are given a parachute whose parachute chain is a link chain it is easy to construct a corresponding umbrella. Alternatively, we can construct a twin-mast: pick two consecutive link vertices v_i, v_{i+1} along the link chain and make a rectangle consisting of $\{v_1, v_i, v_{i+1}, v_n\}$. Place u below the two diagonals thereby preventing v_i from seeing v_n and v_{i+1} from seeing v_1. Refer to Fig. 2. Next, position $(v_2, \ldots, v_{i-1})$ along an arc A_1 out of sight of v_{i+1}. Similarly, position $(v_{i+2}, \ldots, v_{n-1})$ along an arc A_2 out of sight of v_i. It is straightforward to check that the resulting polygon is a twin-mast. A single-mast can be constructed in a similar fashion.

We therefore have the following result.

Theorem 3.4 *A graph G with Hamiltonian cycle H is the visibility graph of an umbrella or a single- or twin-mast if and only if it is a parachute whose parachute chain is a link chain.*

We now move on to parachutes whose parachute chain is a private-link chain. We must modify the above constructions to allow private vertices.

An *extended-umbrella*, shown in Fig. 3, is an extension of an umbrella where we may insert between two consecutive vertices l_i and l_{i+1} of the reflex chain of the umbrella, a convex chain of vertices which are not visible from any reflex vertex other than l_i and l_{i+1}. We can do this by inserting the convex chain inside a pocket determined by the wedge u, l_i, l_{i+1}. See Fig. 3.

The same idea works for extended-masts, either twin or single, where convex chains reside in pockets determined by two consecutive vertices taken from $(v_1, \ldots, v_n)$. See Fig. 4. Care must be taken for the consecutive pair v_i, v_{i+1}, and, in case they are consecutive, for the pairs v_1, v_i and v_{i+1}, v_n. We obtain:

Theorem 3.5 *A graph G with Hamiltonian cycle H is the visibility graph of an extended-umbrella or an extended single- or twin-mast if and only if it is a parachute whose parachute chain is a private-link chain.*

3.2 Parachutes That Are Knots

In this subsection we show that a parachute that is a visibility graph has at most one knot, and prove that this knot must have a special form.

Lemma 3.6 *A knot can only be realized, if at all, as the visibility graph of a convex chain.*

PROOF For any three consecutive vertices v_i, v_{i+1}, v_{i+2} in a knot, there is an edge (v_i, v_{i+2}), because every clique in the knot has at least three vertices. □

It is easy to show that every knot has at least one pair of non-adjacent vertices. Ghosh's conditions constrain the blocking vertex:

Lemma 3.7 *Only the universal vertex may be a blocking vertex for a pair of non-adjacent vertices in a knot in a parachute.*

PROOF Let v_i, v_j be the non-adjacent pair, with $i < j$. This pair cannot be blocked by a vertex between v_i and v_j because they are all part of the knot, which can only be realized as a convex chain.

If there is a blocking vertex v_k with k not between i and j, then no vertex between v_j and v_k in the Hamiltonian cycle can be adjacent to any vertex between v_k and v_i in the Hamiltonian cycle. But the universal vertex must be in one of these two chains. Thus only the universal vertex can block. □

Lemma 3.8 *If a parachute contains two or more knots, it is not a visibility graph.*

PROOF We will use Ghosh's conditions to show that the universal vertex cannot block two knots. Let K_1, K_2 be two knots of the parachute G with Hamiltonian cycle H. Assume without loss of generality that the vertices of K_1 precede the vertices of K_2 along H (they may have one vertex in common). As observed earlier, there exist two pairs of non-adjacent vertices (v_1, w_1) in K_1 and (v_2, w_2) in K_2, $v_1 \prec w_1 \preceq v_2 \prec w_2$. By Lemma 3.7, each of these pairs can only be blocked by u, the universal vertex. However, note that the two pairs are separable with respect to u. Thus by Gosh's third condition, (G, H) is not a visibility graph of a simple polygon. □

If K is a knot in a parachute, and the cliques of K are $c_i, \ldots, c_j$ then a vertex of K that is neither in c_i nor c_j is called an *internal vertex* of the knot.

Lemma 3.9 *If a parachute is a visibility graph then it cannot have a knot with an internal vertex.*

PROOF We apply Ghosh's conditions once more. Suppose G with Hamiltonian cycle H is a parachute with a knot K, consisting of cliques $c_i, \ldots, c_j$, that has an internal vertex v. There exist vertices $x \in c_i$ and $y \in c_j$, such that v is adjacent to neither x nor y. By Lemma 3.7 the non-adjacent pairs (x, v) and (v, y) have u as their sole blocking vertex. But the two pairs are separable with respect to u, so by Ghosh's third condition (G, H) is not a visibility graph. □

We will now consider the case of a clique interval system consisting of a single knot (with no internal interval) plus one universal interval. We will show that this corresponds to a visibility graph, by applying the characterization of visibility graphs of spiral polygons given by Everett and Corneil [EC]. From their result we will get a particular form of spiral polygon which we will use later when we allow the clique interval system to be more than a knot.

Lemma 3.10 *Suppose we have a clique interval system whose intervals consist of a knot with no internal interval plus one universal interval. Then the corresponding interval graph is a visibility graph of a spiral polygon with exactly one reflex vertex.*

PROOF We will apply the results of Everett and Corneil [EC] who gave necessary and sufficient conditions for an interval graph G with a specified Hamiltonian cycle H to be a visibility graph of a spiral polygon, with H as the polygon boundary. If the Hamiltonian cycle H is $(x, r_1, \ldots, r_j, y, l_k, l_{k-1}, \ldots, l_1, x)$ where $R = (x, r_1, \ldots, r_j, y)$ is the proposed reflex chain, and $L = (y, l_k, \ldots, l_1, x)$ is the proposed convex chain, their conditions are that:

1. *R spans G.*
2. L is *straight*.
3. G has no *outlying vertices*.

In our case x and y are private vertices in the first and last clique, respectively; j is 1 and r_1 is the universal vertex u; and $l_1, \ldots, l_k$ are the remaining vertices in progressive order.

The condition that R spans G means that x and y are in the first and last cliques, respectively, each vertex in $r_1, \ldots, r_j$ is in at least two cliques, and for each pair of consecutive cliques, c, c', there is exactly one vertex r_i in both c and c'. In our case this condition is satisfied, because r_1, being a universal vertex, is in all cliques.

The condition that L is straight is precisely the condition that $l_1, \ldots, l_k$ is a progressive ordering. Thus this condition holds in our case.

Finally, an outlying vertex is a vertex not in R whose interval is dominated by another interval. In our case, the fact that the knot consists of all the cliques, and has no internal intervals means that the graph has no outlying vertices. Thus all three of the Everett-Corneil conditions are met, and our graph is the visibility graph of a spiral polygon with reflex chain (x, u, y)—in other words with a single reflex vertex. □

3.3 Parachutes With Knots

We are ready for our main theorem:

Theorem 3.11 *A parachute is a visibility graph iff it has at most one knot and that knot has no internal vertices.*

The necessity of these conditions has been established in the previous subsection. It remains to construct a polygon whose visibility graph is a parachute with at most one knot and no internal vertex in that knot. We first examine the structure surrounding a knot in a parachute.

Lemma 3.12 *Let K be a knot consisting of cliques $c_i, \ldots, c_j$. The only possible vertices in K but also in cliques outside K are one link vertex in $c_{i-1} \cap c_i$ and one link vertex in $c_j \cap c_{j+1}$.*

PROOF Follows from Lemmas 3.1 and 3.2. □

By Lemma 3.10 the visibility graph of a parachute consisting of a single knot and a universal vertex is a spiral polygon with a single reflex vertex. Such a spiral S can be constructed by the algorithm in [EC]. Assume $(v_l, \ldots, v_r)$ are the vertices of the knot. Since S is star-shaped from u, the wedge determined by v_l, u, v_r is empty (Fig. 5).

As we move from v_l to v_r, let v and v' be the last vertex visible from v_l and the first vertex visible from v_r respectively. Let W_r be the open wedge determined by the two lines through u, v_r and v', v_r. See Fig. 6. Points in W_r are hidden from all vertices except u and v_r. Similarly let W_l be the open wedge determined by the two lines through u, v_l and v, v_l.

By Lemma 3.12, chains $(v_1, \ldots, v_{l-1})$ and $(v_{r+1}, \ldots, v_n)$ are private-link chains. First suppose that these chains are actually link chains. Fix a point v_1 in W_l. See Fig. 7. Join v_1 to u. Since v_1 is in the interior of W_l and u sees edge (v_l, v_1), we can draw a circular arc A passing through v_1 and v_l visible from u in the interior of triangle $\Delta u, v_l, v_1$ and inside W_l. Place $(v_2, \ldots, v_{l-1})$ along A. Do the same in W_r for $(v_{r+1}, \ldots, v_n)$.

If either $(v_1, \ldots, v_{l-1})$ or $(v_{l+1}, \ldots, v_n)$ is actually a private-link chain, replace the edges along the reflex chain by convex chains in pockets as described in section 3.1.

4 Further Work

More generally, we can begin to deal with any interval graph with a Hamiltonian cycle that can be divided into two progressive chains. (The present work assumed that one chain had just one vertex.) We divide each progressive chain into *trusses* and *flexes*, structures that slightly generalize knots and private-link chains. In this general case Ghosh's conditions are not sufficient to guarantee a visibility graph—we have a necessary condition that must be added.

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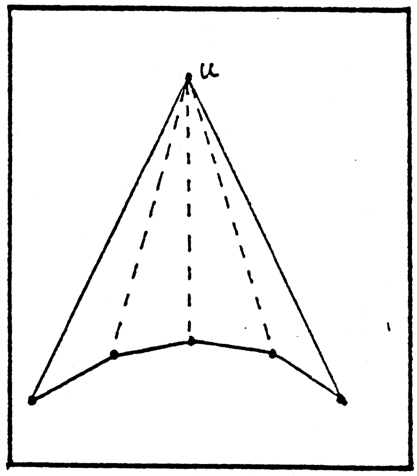

Fig. 1

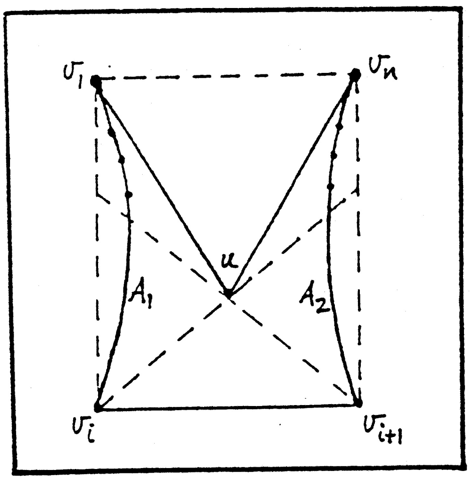

Fig. 2

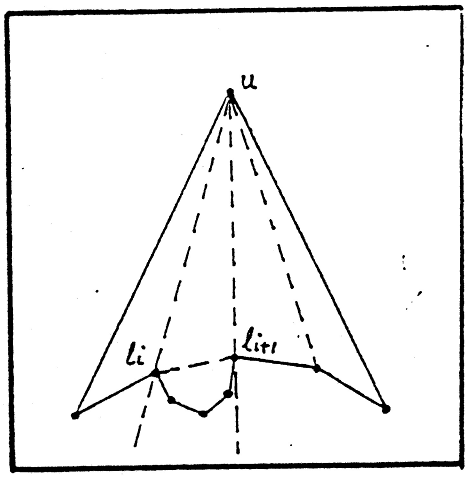

Fig. 3

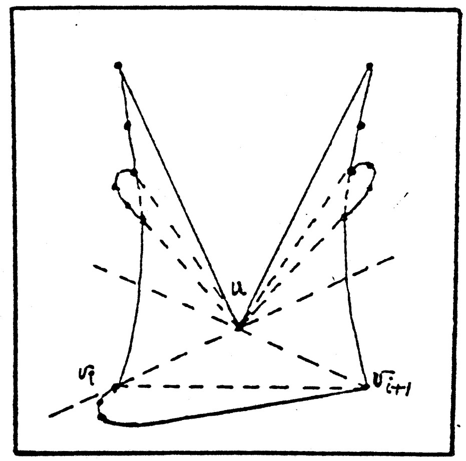

Fig. 4

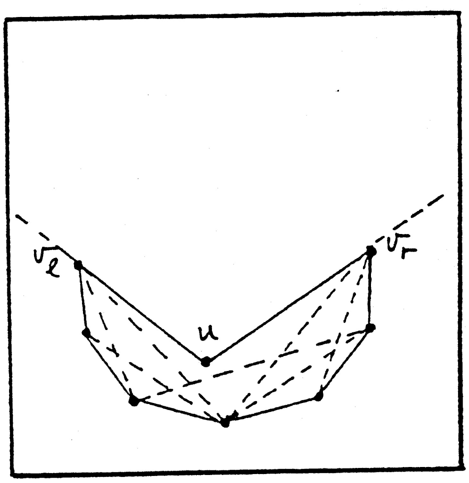

Fig. 5

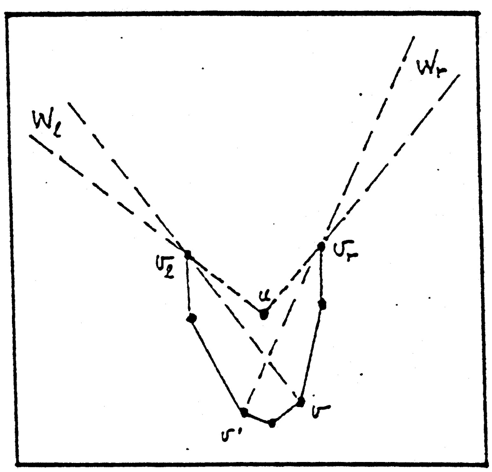

Fig. 6

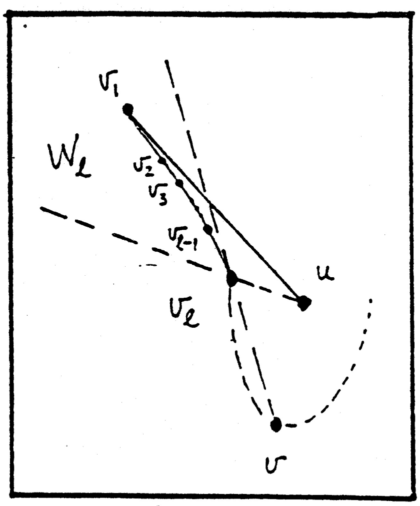

Fig. 7