The Lawnmower Problem

(Extended Abstract)

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Abstract

We discuss the *Lawnmower Problem*: Given a polygonal region, find the shortest closed path along which we have to move a given object (typically a square or a circle), such that any point of the region will be covered by the object for some position of it movement. In another version of the problem, known as the *Milling Problem*, the object has to stay within the region at all times.

Practical motivations for considering the Lawnmower Problem come from manufacturing (spray painting, quality control), geography (aerial surveys), optimization (tour planning for a large number of clients with limited mobility), and gardening. The Milling Problem has gained attention by its importance for NC pocket machining.

We show that both problems are NP-hard and give the (to our knowledge) first proof of a constant approximation factor for an approximation algorithm.

1 Introduction

Anybody who has ever mowed a lawn has been confronted with the situation shown in Figure 1: For a given region covered by grass, find a short path along which to move a lawnmower, such that all the grass is cut.

This *Lawnmower Problem* arises in many practical situations. Motivations from manufacturing deal with production and quality control:

- How do we have to move the nozzle of a spray painting device in order to coat the whole surface of an object?
- How do we have to move the sensor of a detector that checks objects for imperfections?

A question of similar type arises from geographic surveys:

- How do we have to move a video camera or other detector to do a complete survey of a given region?

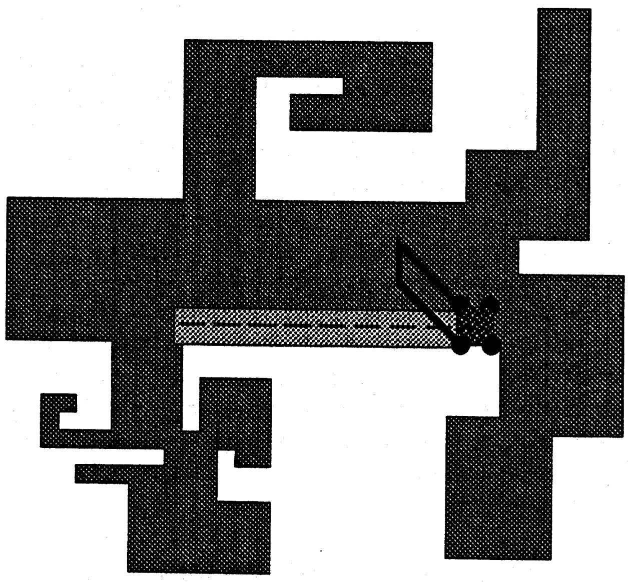

Figure 1: The Lawnmower Problem

We should also point out that the Lawnmower Problem is closely related to the Traveling Salesman Problem with mobile clients:

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- What is the shortest tour for a salesman who has to visit a number of clients, each of which is willing to travel some bounded distance in order to meet the salesman?

Instead of considering the region of mobility for each client, we can think of moving this region with the salesman – thereby getting the Lawnmower Problem with the set of client locations being the set that has to be "mowed". For a finite set of points, this problem has been studied by Arkin and Hassin [1]. Our results allow us to consider the case of an infinite number of clients with original positions within a given region.

Closely related to TSP problems are "watchman route" problems, which ask one to find a short tour so that a mobile watchman sees all points within a given domain. Our lawnmower problem is most closely related to the watchman route *with limited visibility* problem, as studied by Ntafos [9], who introduces the "d-sweeper problem": How does one sweep a polygonal floor with a circular broom of radius d so that the total travel of the broom is minimized? By studying the problem of approximating TSP tours on *simple* grid graphs, Ntafos gives an approximation algorithm (within factor 1.33) for the d-sweeper problem in a simple polygon, *provided* that d is sufficiently "small" in comparison with the dimensions of the polygon.

A question of slightly different type is motivated by the industrial process of *pocket machining*: From a given workpiece, material has to be removed by means of milling; the resulting negative region is called a "pocket". The difference to the Lawnmower Problem is that the carving object may not leave the given region.

Geometric aspects of pocket machining have been studied extensively, most notably by Held [6]. He gives a survey of practical aspects and implementation, with the main emphasis being on achieving feasibility of a milling tour. Aspects of complexity are only mentioned very briefly: The question of polynomiality or NP-hardness of the Milling Problem and some of its special aspects is stated as an interesting problem for further theoretical research. (Aspects of approximating the optimal path length are not mentioned.) Held also stresses the necessity for future research of the problem with a more theoretical orientation. In this first paper, we try to shed light on some theoretical aspects dealing with the optimization of the length of a milling and lawnmower tours.

The main results derived in this paper are as follows:

- It is NP-hard to determine an optimal lawnmower tour, even if the given region is a simple polygon.
- It is NP-hard to determine an optimal milling tour if the given region is a polygon with holes.
- The shortest lawnmower tour can be approximated within a constant factor.
- The shortest milling tour can be approximated within a constant factor.

The rest of the paper is organized as follows. Section 2 discusses a basic problem of the description of feasible lawnmower tours. Section 3 gives an NP-hardness proof for the Lawnmower Problem and the Milling Problem. Section 4 and Section 5 describe provable approximation factors for the Lawnmower Problem and the Milling Problem. Section 6 discusses further research.

2 Describing lawnmower tours

In general, we will assume that our region is described as a polygon (possibly with holes) with n edges. However, even for very simple polygons the number of edges in any feasible lawnmower tour may be exponential in the size of the input data – see Figure 2 for an easy example.

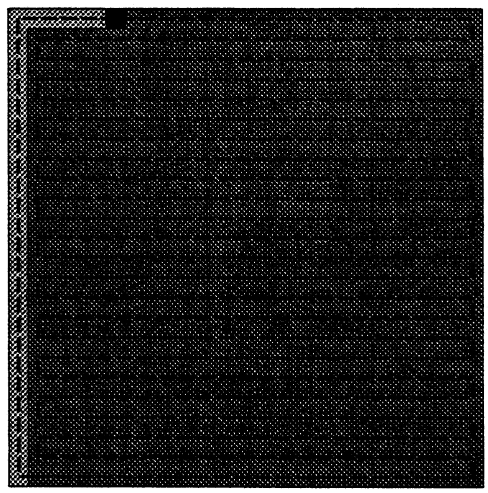

Figure 2: A feasible lawnmower tour may require a large number of edges

This problem can be solved in the following two ways:

1. We consider the ***value of the input numbers*** and not the ***length of the input numbers*** as the basic measure of complexity. This means we consider methods that are ***pseudopolynomial*** in the usual sense.
2. We consider tours that consist of a polynomial number of pieces with a regular structure, meaning that we could encode a tour in a shorter way than by giving a list of its edges.

In his book on the Milling Problem [6], Held concentrates on two natural strategies – see Figure 3: (1) contour-parallel milling, and (2) axis-parallel milling.

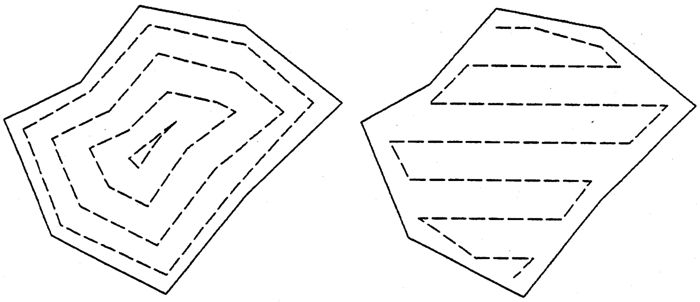

Figure 3: Contour-parallel and axis-parallel milling

It seems reasonable to assume that partial tours that follow one of these two strategies can be encoded efficiently.

3 NP-hardness results

In this section, we will show that the Lawnmower Problem is NP-hard. We will outline the proof for the case where the mowing object is a unit square; the proof for the case of a circle follows with a simple modification. It also follows from our proof that the Milling Problem is NP-hard.

Theorem 3.1 *The Lawnmower Problem for a mowing square is NP-hard.*

Proof: Our proof makes use of the reduction of the problem Hamiltonian Circuit in Grid Graphs from Hamiltonian Circuit in Planar Bipartite Graphs with Maximum Degree 3, as described by Johnson and Papadimitriou [8]. (See also Itai, Papadimitriou and Swarcfiter [7].) In a first step, a planar bipartite graph G with n vertices (each of maximum degree 3) is represented by a grid graph $\overline{G}$, such that $\overline{G}$ has a Hamiltonian circuit if and only if G has a Hamiltonian circuit. (See Figures 4 and 5 for an example of such a representation.)

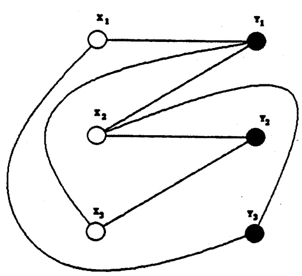

Figure 4: A planar bipartite graph

From this grid graph $\overline{G}$, we construct a polygonal region P as follows: At each of the $m = O(n)$ grid vertices

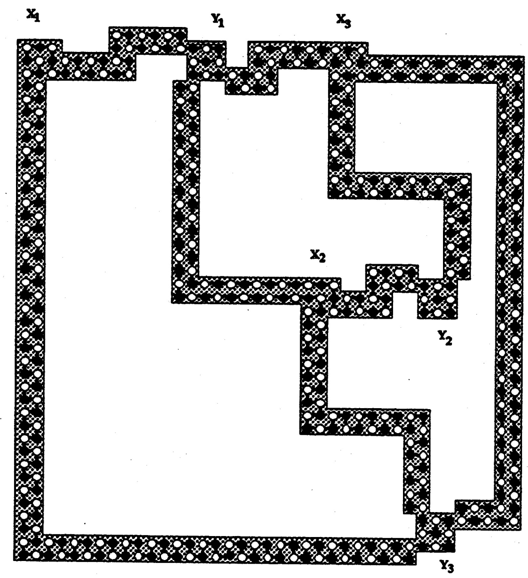

Figure 5: NP-hardness of the Lawnmower Problem

in $\overline{G}$, place the center of a unit square. P is the union of all these unit squares.

It is easy to see that the existence of a tour of length m on the grid vertices describing $\overline{G}$ implies the existence of a lawnmower tour of length m. On the other hand, it is relatively straightforward to show from the special structure of the constructed region that a lawnmower tour of length m induces a tour of length at most m in the grid graph. (See Figure 5.) □

Corollary 3.2 *The Lawnmower Problem is NP-hard even in the case in which the region to be mowed is a simple polygon.*

Proof: The idea is to connect holes in the polygon by sufficiently narrow "cuts", as indicated in Figure 6.

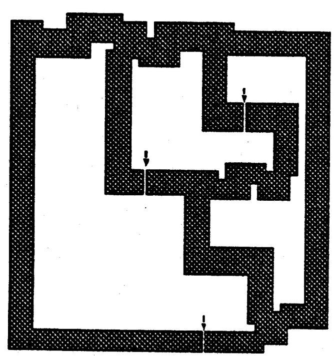

Figure 6: NP-hardness of the Lawnmower Problem for simple polygons

Clearly, the existence of cuts of arbitrarily small area does not change the relation between a short lawnmower tour and a short Hamiltonian circuit in the grid graph. □

Since the optimal lawnmower tours in Theorem 3.1 are also feasible milling tours, we get

Corollary 3.3 *The Milling Problem for unit squares and polygons with holes is NP-hard.*

It should be noted that a similar approach for showing NP-hardness of the Milling Problem in the case of *simple* polygons may not be helpful: It is still unknown whether the existence of a Hamiltonian circuit in a grid graph without holes can be decided in polynomial time.

We conclude this section by pointing out that the Lawnmower Problem for a circular mower is NP-hard:

Theorem 3.4 *The Lawnmower Problem for the case of a circular mower is NP-hard.*

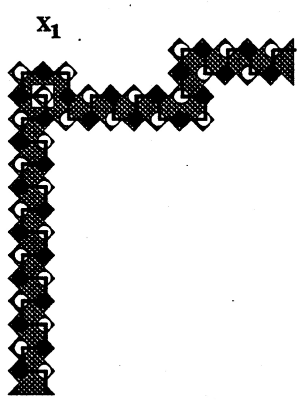

Figure 7: NP-hardness for a circular lawnmower

Sketch: The basic idea is similar to Theorem 3.1. At each of the grid vertices, we place a diamond of diameter 1 instead of a unit square; furthermore, we place additional triangles of a quarter size of the diamond, depending on the neighbors in the quadrant. (See Figure 7.) We omit further details at this point. □

4 An approximation method for the Lawnmower Problem

In the following, the term "pixel" refers to a unit square with vertices being integer grid points.

Theorem 4.1 *Let R be a connected region of grass in the plane, and let N denote the number of integer lattice pixels that intersect R. Assume that the mowing object is a unit square, required to remain aligned with the coordinate axes. Then in time $O(N)$ one can compute a lawnmower tour for R whose length is at most 6 times the length of an optimal lawnmower tour restricted to axis-parallel movement. Without the axis-parallel restriction, the approximation factor becomes $5\sqrt{2}$.*

Proof: Let P denote the set of all pixels that contain some grass. Since R is connected, so is the grid graph G whose nodes correspond to centers of pixels of P.

Our approximation algorithm is simple: Obtain a tour by doubling a spanning tree of G. This can clearly be done in time linear in the size of G — namely in time $O(N)$.

The feasibility of the tour produced is clear from the fact that it will necessarily cover all pixels of P, and hence will mow all of R.

The length of a spanning tree in G is simply $N-1$, and the length of the tour produced is therefore at most $2(N-1)$. On the other hand, by Lemma 4.2 below,

$$N \leq 3L_{OPT} + 9,$$

where L_{OPT} is the length of an optimal lawnmower tour. Thus, we obtain a bound on the length, L_{APP}, of our approximating tour:

$$L_{APP} \leq 2(N-1) \leq 6L_{OPT} + 16$$

□

Lemma 4.2 *If a unit square is moved without rotation in the plane, such that its center moves a total distance of L, then the region swept out can intersect at most $5/\sqrt{2}L + 9$ of the pixels defined by the integer lattice points. If the motion of the square is further restricted so that its translation is always axis-parallel, then at most $3L + 9$ pixels are intersected by the swept region.*

We can prove Lemma 4.2 by estimating the number of grid points contained in the region swept by a square of edge length 2 while moving it along a path of length L. (See Figure 8.) We omit further details here.

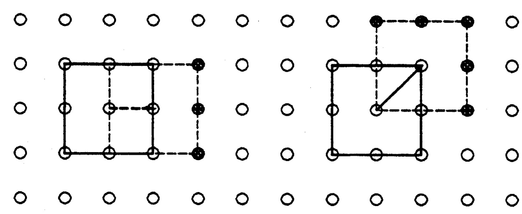

Figure 8: Counting the number of intersected pixels

Theorem 4.3 *Let R be a polygonal region of grass consisting of $k < \infty$ connected components having a total of n vertices, and let N denote the number of integer lattice pixels that intersect R. Assume that the mowing object is a unit square, required to remain aligned with the coordinate axes. Then in time $O(N + n \log n)$ one can compute a lawnmower tour for R whose length is at most a constant times the length of an optimal lawnmower tour.*

Sketch: Let P denote the set of pixels that contain some grass. Let G be the grid graph corresponding to the center points of pixels P. Then G may have many connected components. Let k_G denote the number of components; note that $k_G \leq k$. Within each connected component, C_i, of G, we can proceed as before, and obtain a lawnmower tour approximation within factor 6 or $5\sqrt{2}$ of optimal.

Now, for each component C_i we "fatten" the grass region that lies within C_i by an amount 1/2 (in L_∞ metric), obtaining a polygonal set Q_i. Treating the sets Q_i as regions through which travel is "free", we construct a minimum spanning tree on the sets Q_i (in time $O(n \log n)$, using either an L_1 or a Euclidean minimum spanning tree algorithm, depending on whether or not we are restricted to axis parallel movements). By increasing each edge of the MST by at most length 2, we can make attachments between center points of pixels of the components C_i. Then, we can construct one lawnmower tour that mows all of R by concatenating each of the approximating tours for each C_i, together with a doubling of the extended edges minimum spanning tree.

The length of the extended minimum spanning tree is bounded above by the length L_{OPT} of the overall optimal tour *plus* $2(k_G - 1)$ (the total extension to all edges of the MST). Also, for each connected component we know that $L_{APP}(C_i) \leq cL_{OPT}(C_i)$, where c is either 6 or $5\sqrt{2}$. Finally, one can show that Lemma 4.2 implies that $k_G \leq 3L_{OPT} + 9$. Thus, our overall bound becomes

$$\begin{array}{lccc} & L_{APP} & & \\ \leq & \sum_i L_{APP}(C_i) & + & 2L_{MST} \\ \leq & c\sum_i L_{OPT}(C_i) & + & 2L_{OPT} + 2(2(k_G-1)) \\ \leq & (c+2)L_{OPT} & + & 4(k_G-1) \\ \leq & (c+14)L_{OPT} + 32. & & \end{array}$$

□

The methods described in Theorem 4.1 and Theorem 4.3 can be used for constructing an approximating lawnmower tour that has a fast running time in an even stricter sense:

Corollary 4.4 *We can construct an approximating tour as in Theorem 4.1 or Theorem 4.3 in time $O(nb)$ or $O(nb \log n)$, where n is the number of edges of the region and b is the complexity of describing a spanning tree of the grid points in a convex quadrilateral with two horizontal edges.*

Sketch: We can construct a horizontal trapezoidization of the polygonal region R in linear time (see [2]), which gives us a partitioning of the vertex set of G according to the containing trapezoids. Within each such trapezoid, the spanning tree for G has a regular structure, as discussed in Section 2, whose description is of complexity b. We get the claimed complexity by linking together these partial spanning trees, and proceeding as before. □

We only state without further elaboration that the methods described above can also be used for finding an approximation in the case where the mowing object is a unit disk instead of a unit square. Instead of the cover by unit squares that is based on the orthogonal grid, we consider the cover by unit disks based on the hexagonal grid. (See Figure 9.)

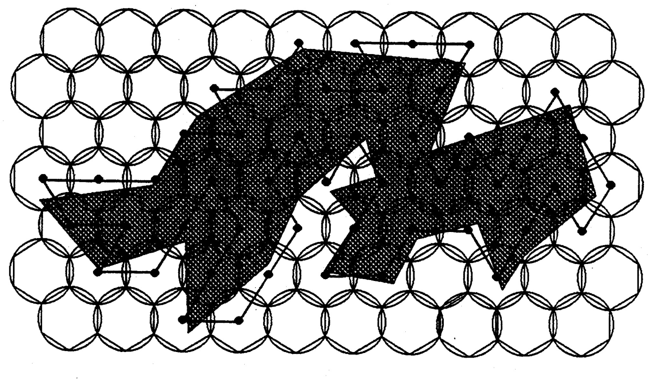

Figure 9: Approximation for the case of a mowing disk.

Theorem 4.5 *Let R be a connected region of grass in the plane, and let N denote the number of hexagonal lattice pixels that intersect R. Assume that the mowing object is a unit disk, required to remain aligned with the coordinate axes. Then in time $O(N)$ one can compute a lawnmower tour for R whose length is at most $4\sqrt{3}$ times the length of an optimal lawnmower tour.*

5 An approximation method for the Milling Problem

We turn our attention now to the case of milling tours, in which the cutting tool is required to stay within the region R that is to be milled. Of course, not all regions R are millable (while it is true that all regions R can be mowed).

In this case, our strategy changes slightly, but we are still able to obtain a constant-factor approximation result:

Theorem 5.1 *In time $O(n \log n)$, one can decide whether a region with n sides (straight or circular arc) can be milled by a unit disk or unit square, and, within the same time bound, one can construct a tour of length at most 3 times the length of an optimal milling tour.*

Proof: (Sketch) In a first step, using a Voronoi diagram of R (in either Euclidean or L_∞ metric), we construct the set $B \subset R$ of all points within R that are feasible placements for the center point of the milling cutter. It is easy to check whether B is connected and whether each point of the boundary ∂B can be reached

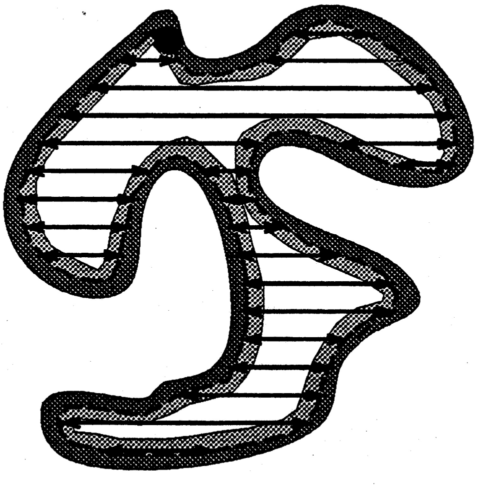

Figure 10: Approximating a milling tour

by placing the center of the milling object in B. (Clearly, no polygon can be milled by a unit disk and no polygon with acute angles can be milled by a unit square. For milling with a unit disk, we need to consider regions R with curved boundaries, e.g. circular arcs, that have curvature at most 1.)

Assume B is connected and all points of the boundary can be reached, so that R can be milled. Then the length $L_{\partial B}$ of the boundary ∂B of B is a lower bound for length L_{OPT} of an optimal milling tour. We write $R_{\partial B}$ for the region milled by moving along ∂B. If $R_{int} := R \setminus R_{\partial B}$ is nonempty, we can cover it by a set of s horizontal strips S_i of vertical width 1 and disjoint interior, as shown in Figure 10. Since we need at least the length $L_{str} = \sum_{i=1}^{s} L_{S_i}$ to mill R_{int}, we conclude that $L_{str} \leq L_{OPT}$.

There is a feasible milling tour of length $L_{\partial B} + 2L_{str}$: Follow ∂B; whenever encountering a strip that is not yet milled, include it in the tour by running through it "back and forth". (See Figure 10.) We conclude that we get a milling tour no longer than $3L_{OPT}$. □

As before, we can argue that our approximation method is polynomial in an even stricter sense, if we take into account that the structure of the constructed tours is simple enough to allow compact encoding.

6 Further research

It is expected that the approximation factors described in the previous section can be improved. However, any approximation method has to deal with the inherent difficulties of the size of a feasible solution, as we described in Section 2. This may exclude solutions of a fundamentally different type.

It may be interesting to consider a combination of the Lawnmower Problem and the Milling Problem: The complement of the region R is subdivided in parts that may be crossed by the lawnmower, and "forbidden regions" that may not be touched. (This situation is familiar to anyone who has had to deal with lawn, pavement and flower beds.)

Another important consideration from a practical point of view is the shape of a feasible path: If a path has too many turns, it may require a slower processing speed, thus spoiling the benefits of a short tour. It seems worthwhile to examine aspects of the link distance necessary to "mow" a given polygon. Another consideration of similar type is the size of the angles at turns – see Held [6]. For a finite set of points, the existence of such "angle-restricted tours" has been studied by Fekete and Woeginger [4].

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