How to Look Around a Corner

Extended Abstract

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Abstract

We consider a problem of motion planning under uncertainty. A robot can navigate freely in the plane and, using a built-in vision system, can determine distances and angles. Initially, the robot stands at a point close to a an edge of a polygonal obstacle (e. g. a wall of a huge building) and faces a corner at distance 1 from its position. The other wall which forms the corner is invisible from the starting position and the robot does not know the angle of the corner. The task of the robot is to move on a short path to a point where that wall becomes visible.

We show that there is a competitive strategy which guarantees that, for any possible value of the angle, the length of the path the robot walks until it can look around the corner is bounded by the length of the shortest path to do so, times the constant $c \approx 1.21$. Furthermore, we prove that our strategy is optimal in that no smaller competitive factor than c can be achieved. We give a simple formula for the robot to find the optimal path.

Key words. Motion planning, navigation, competitive algorithms, uncertainty, robotics.

1 Introduction

Algorithmic motion-planning in robotics is a classical field in computational geometry, see Schwartz and Sharir [10], Schwartz and Yap [11], or Mitchell [7] for surveys.

In the majority of the existing work it is assumed that the environment in which the system moves is known in advance. In real life, this assumption is not always granted. Autonomous vehicles should be able to find their ways through, or learn, unknown terrain as efficently as possible. This means that the task must be accomplished correctly, but as little as possible of resources like time or energy should be used.

In the last years, several researchers independently began to apply to geometric planning problems the concept of *competitive algorithms* introduced by Sleator and Tarjan [12]. Here one compares what can be achieved with incomplete information against what could be achieved if full information was available. More precisely, S is a *competitive strategy* for problem class $\mathcal{P}$ if there exists a constant c such that, for each instance P of $\mathcal{P}$, the cost of applying S to P does not exceed c times the cost of solving P in an optimal way, given full information. The minimum c satisfying this condition is called the *competitive factor* of S.

Among other work, competitive geometric algorithms have been developed by Papadimitriou and Yanakakis [9], Blum, Raghhavan, and Schieber [1], and Eades, Lin, and Wormald [4] for path planning in the presence of obstacles in the plane, by Deng, Kameda, and Papadimitriou [3] for learning the interior of a polygon that may have a bounded number of holes, and by Klein [6] for finding a path in the interior of special simple polygons called streets.

In Section 2 we define and characterize competitive strategies for the corner problem. In Section 3 we solve the optimality problem. Our approach leads to a differential equation of which a closed-form solution is apparently not provided by the theory, but we can show that the required solution must exist.

From the bare existence, and from the functional properties of the differential equation, we are able to derive that the solution of the above differential equation leads to a competitive strategy whose factor equals 1.21218..., and that no better strategy exists. A main step is in proving that the curve implied by this strategy is convex.

While the analysis of our strategy and the proof for optimality are rather complicated and use means from the theory of ordinary differen-

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tial equations, the resulting strategy is surprisingly simple.

We present a generalization of the problem in Section 4.

2 Preliminaries

In this paper we study an elementary problem related to learning an unknown environment. Suppose that two halflines meet at the origin O, as shown in Figure 1. The shaded wedge formed by the halflines is opaque; it could be the corner of a huge, non-rectangular building. Now assume that on one of the halflines a mobile robot is located at point W, outside the wedge, that is equipped with an on-board vision system facing O. Its task is to inspect the other halfline which is invisible from W (but for its endpoint, O).

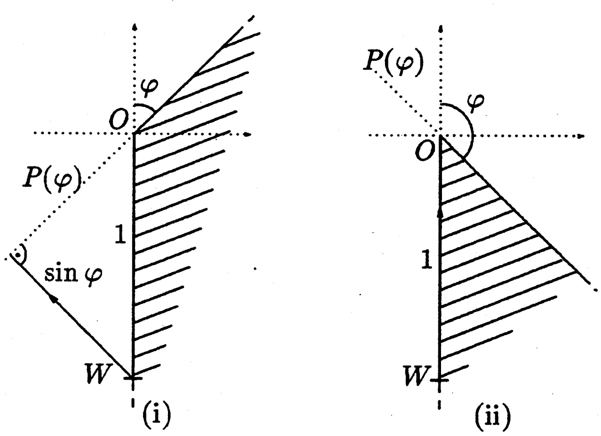

Figure 1: The shortest path from W to $P(\varphi)$.

Given full information about the angle of the corner, the robot should obviously move along the shortest path to $P(\varphi)$, the prolongation of the invisible halfline. Here, $\varphi \in [0, \pi]$ denotes the angle between the invisible halfline and the prolongation of the visible one.

If the value of φ lies between 0° and 90° then the shortest path from W to $P(\varphi)$ is the perpendicular; see Figure 1 (i). If φ is bigger than 90° then the point of $P(\varphi)$ closest to W is the corner itself; see (ii).

By $a(\varphi)$ we denote the distance between W and $P(\varphi)$, we have

$$a(\varphi) = \begin{cases} \sin\varphi & : \quad 0 \le \varphi \le \frac{\pi}{2} \\ 1 & : \quad \frac{\pi}{2} < \varphi \le \pi \end{cases}$$

Note that a is continuously differentiable.

But the robot does not know the actual value of φ. So, how should it walk? Obviously, walking straight to the corner fulfills the task. But the length of the path created is 1, whereas an arbitrarily short path could suffice for small values of φ. In fact, walking straight in any fixed direction does *not* lead to a competitive solution.

A *strategy* for our problem should be a curve that starts at point W on the visible wall and leads to the prolongation of the visible wall. Since a strategy can simply be shortened if its intersection with a line through O contains more than one point, we allow only curves which can be described with the help of a function s of the angle φ as follows.

Definition 1 A curve $S = \big(\varphi, s(\varphi)\big)$ in polar coordinates about O is called a *strategy* for the corner problem if the following holds.

(i) s is a continuous function on an interval $[0, \sigma]$, where $\sigma \le \pi$.

(ii) On the open interval $(0, \sigma)$, s is piecewise continuously differentiable and $s'(0)$ exists (possibly $\pm\infty$).

(iii) $s(0) = 1$.

(iv) If $s(\sigma) \ne 0$, then $\sigma = \pi$.

The last property states that S must arrive at $P(\pi)$, including the corner.

Let $A_S(\varphi)$ be the length of the path generated by strategy S up to the angle φ. The *competitive function*, $f_S(\varphi)$, of S is the ratio of $A_S(\varphi)$ and $a(\varphi)$, and its *competitive factor*, c_S, is the maximum value of $f_S(\varphi)$.

$$\begin{aligned} f_S(\varphi) &= \frac{A_S(\varphi)}{a(\varphi)} \\ c_S &= \sup_{\varphi \in (0,\pi]} f_S(\varphi) \end{aligned}$$

By $f_S(0)$ we mean $\lim_{\varphi \to 0} f_S(\varphi)$, if it exists. The problem is to find a strategy whose competive factor is as small as possible.

First, we show that each sensible strategy is in fact competitive.

Lemma 2 *Let $S = \big(\varphi, s(\varphi)\big)$ be a strategy. Then S is competitive iff $|s'(0)| < \infty$. The estimation*

$$c_S \geq \sqrt{s'^2(0) + 1}$$

holds for the competitive factor.

Proof. Since

$$A_S(\varphi) = \int_0^\varphi \sqrt{s'^2(t) + s^2(t)}\,dt \qquad (1)$$

holds for the arc length of a curve in polar coordinates, we obtain from de l'Hospital's theorem (i. e. by taking derivatives in both numerator and denominator)

$$\begin{aligned} c_S &\geq f_S(0) \\ &= \lim_{\varphi \to 0} \frac{A_S(\varphi)}{\sin\varphi} \\ &= \lim_{\varphi \to 0} \frac{\sqrt{s(\varphi)^2 + s'^2(\varphi)}}{\cos\varphi} \\ &= \sqrt{s'^2(0) + 1} \end{aligned}$$

Since f_S is a continuous function on the interval $[0, \sigma]$, it takes on its maximum value. □

To give an example, consider strategy S_1 that walks along the circle through W with center in the corner, i. e. $s_1(\varphi) = 1$ for all φ, see Figure 2. We have $A_{S_1}(\varphi) = \varphi$ for all φ, and $f_{S_1}(\varphi) = \frac{\varphi}{\sin\varphi}$ for $\varphi \in [0, \frac{\pi}{2}]$ and $f_{S_1}(\varphi) = \varphi$ for $\varphi \in [\frac{\pi}{2}, \pi]$. It is easy to check that f_{S_1} attains its maximum at $\varphi = \pi$, thus $c_{S_1} = \pi \approx 3.14159$.

A better strategy, S_2, is the following. We walk along the circle with radius $\frac{1}{2}$ through W centered at the mid point between W and O, i. e. $s_2(\varphi) = \cos\varphi$, see also Figure 2. Then $s_2(\frac{\pi}{2}) = 0$, i. e. we reach the corner, so we only need to consider the angles φ in the interval $[0, \frac{\pi}{2}]$.

But $A_{S_2}(\varphi) = \frac{1}{2}(2\varphi)$ holds, implying $f_{S_2}(\varphi) = f_{S_1}(\varphi) = \frac{\varphi}{\sin\varphi}$ for $\varphi \in [0, \frac{\pi}{2}]$. The maximum value c_{S_2} is only $\frac{\pi}{2} \approx 1.57079$.

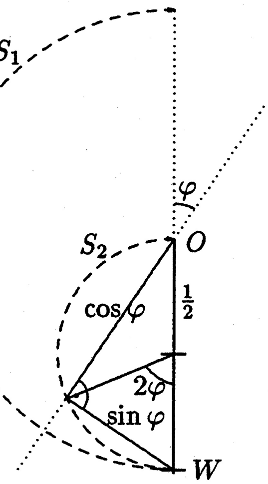

Figure 2: Some simple strategies, achieving competitive factors $c_{S_1} = \pi$ and $c_{S_2} = \frac{\pi}{2}$.

3 A differential equation and the optimal solution

Intuitively, if one tries to improve on a given strategy S by modifying it such that the maximum value for $f_S(\varphi)$ becomes smaller, some other values $f_S(\varphi')$ will increase. The key idea towards an optimal strategy is to assume that this process can reach a state of equilibrium, and to look for a strategy R such that $f_R(\varphi) = c$, i. e. constant, *for all* φ. This constant, c, would then be the competitive factor of the strategy. If there is more than one strategy with this property, we would look for the one with the smallest value of c.

Since $a(\varphi) = \sin\varphi$ for $\varphi \in [0, \frac{\pi}{2}]$, we try to solve the following equation.

$$f_R(\varphi) = \frac{A_R(\varphi)}{\sin\varphi} = c \quad \text{for all } \varphi \in \left[0, \frac{\pi}{2}\right]$$

After inserting Equation 1 of Lemma 2, multiplying by $\sin\varphi$, and taking the derivative with respect to φ, we obtain

$$c\cos\varphi = A'_R(\varphi) = \sqrt{r'^2(\varphi) + r^2(\varphi)}$$

This is an ordinary differential equation for the unknown function r, the initial condition is $r(0) = 1$ because we have to start from W with angle $\varphi = 0$. Since we want the robot to eventually arrive at the corner, the solution should exist on an interval

$[0, \sigma]$ where $r(\sigma) = 0$ holds. For $\varphi \in [0, \sigma)$ the radius $r(\varphi)$ should be strictly positive. We solve the equation for $r'(\varphi)$.

$$r'(\varphi) = -\sqrt{c^2 \cos^2 \varphi - r^2(\varphi)} \tag{2}$$

The negative square root is taken because the solution r should be decreasing, meaning that the robot should always come closer to the corner, as it proceeds.

Equation 2 can be transformed into the following differential equation of Abelian type.

$$w'(x) = \left(w^2(x) + 1\right)\left(1 - w(x) \cot x\right)$$

However, there is no complete theory on equations of this type, and a closed-form solution seems not to be known, see Kamke [5] or Murphy [8]. Nevertheless, the existence of solutions can be shown.

Lemma 3 *There is a constant $c_R > 1$ such that the following holds for solutions of Equation 2 with initial condition $r(0) = 1$.*

(i) If $c = c_R$ then there is a unique solution r which exists on the interval $[0, \pi]$ and $r(\pi) = 0$.

(ii) If $c > c_R$ then there is a unique solution r for each c and it exists on an interval $[0, \sigma]$ where $r(\sigma) = 0$ and $0 < \sigma < \pi$.

(iii) For $|c| < c_R$, there is no solution r such that $r(\sigma) = 0$ for some $\sigma > 0$ or such that r exists on $[0, \pi]$.

The proof involves theorems from calculus and the theory of ordinary differential equations and is too long to be included here.

Now let $R = \left(\varphi, r(\varphi)\right)$ be the strategy in which r is the solution of (2) for $c = c_R$. It now is clear that the competitive factor of R is c_R, and that any other strategy S with constant ratio $f_S(\varphi)$ is worse than that.

The curve R is shown in Figure 3. Using the numerical capabilities of Maple [2], we have determined c_R to be approximately 1.21218.

Since there seems not to exist a closed-form solution of Equation 2, one could think of computing a good numerical approximation of R. But for the robot, there is a much simpler method to find the optimal path.

By introducing the angle α between the tanget to curve R at the actual position and the line from the actual position to the corner, it is easy to eliminate the derivative r' from Equation 2. The result is the formula

$$\alpha = \arcsin \frac{r}{c_R \cos \varphi}$$

which means that the robot can, at each time, calculate its walking direction α if only the current distance to the corner r and the angle φ is known, as well as the constant $c_R \approx 1.21218$. For example, the initial angle for $\varphi = 0°$ and $r = 1$ is $\alpha = \arcsin(\frac{1}{c})$. In particular, the length of the path from the beginning to the actual position needs *not* to be known.

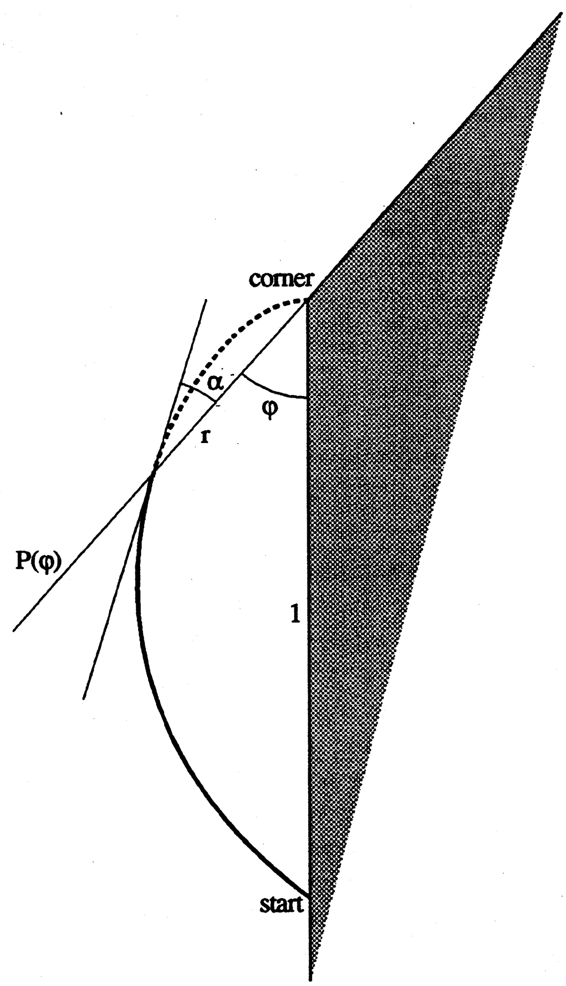

Figure 3: The optimal competitive strategy R.

A nice property of the curve R is its convexity.

Lemma 4 *The strategy R forms a convex curve.*

The proof uses the curvature formula of a curve in polar coordinates and the differential equation which r is a solution of.

Now that the existence of a solution of the differential equation and the convexity of the corresponding curve is established, is is remarkably simple to prove the optimality of R.

Theorem 5 *The strategy R is an optimal competitive strategy for the corner problem.*

Proof.
Let $S = \big(\varphi, s(\varphi)\big)$ be a strategy different from R. We distinguish four cases.

Case 1. If $|s'(0)| = \infty$ then S is not competitive by Lemma 2.

Case 2. If $s'(0) \leq r'(0)$ then we have, again by Lemma 2,

$$\begin{aligned} c_S &\geq \sqrt{s'^2(0)+1} \\ &\geq \sqrt{r'^2(0)+1} \\ &= \sqrt{\left(-\sqrt{c_R^2-1}\right)^2+1} \\ &= c_R \end{aligned}$$

In the remaining cases $s'(0) > r'(0)$ holds. Then there exists an angle ψ such that $s(\varphi) > r(\varphi)$ for $\varphi \in (0, \psi]$.

Case 3. There exists an angle $\chi \leq \frac{\pi}{2}$ such that $s(\chi) = r(\chi)$ and $s(\varphi) > r(\varphi)$ on $(0, \chi)$, see Figure 4. Then $A_S(\chi) > A_R(\chi)$, due to the convexity of R shown in Lemma 4. Hence,

$$c_S \geq f_S(\chi) = \frac{A_S(\chi)}{\sin\chi} > \frac{A_R(\chi)}{\sin\chi} = c_R$$

Case 4. $s(\varphi) > r(\varphi)$ for $\varphi \in (0, \frac{\pi}{2}]$. Then c_S is not less than the total arc length of S, which is bigger than the length of S between $\varphi = 0$ and $\varphi = \frac{\pi}{2}$, plus the length of the line segment from $\big(\frac{\pi}{2}, s(\frac{\pi}{2})\big)$ to the origin. The length of this curve, in turn, is bigger than $A_R(\frac{\pi}{2}) = c_R$, again by convexity of R; see Figure 4. $\square$

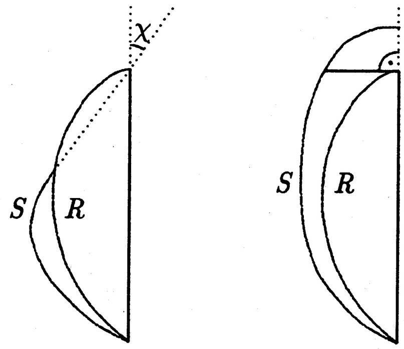

Figure 4: Cases 3 and 4 of Theorem 5.

4 Final remarks

The problem can be generalized to the situation in which the robot's starting point, W, does not lie on a wall but in the free area outside the wedge. The problem becomes different, because now the unknown angle φ can take on its values only in a smaller range.

We are able to construct optimal competitive strategies for all such cases. The optimal curves are still convex, but, surprisingly, they are composed of a solution of a differential equation like before as the first part and a straight line segment perpendicular to the visible wall as the end. The competitive function is constant only in the first part. Its maximum value, the competitive factor, is attained in the constant part and at the end point.

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