Two-dimensional Computation of the Three-dimensional Reachable Region for a Welding Head*

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Abstract

We consider the problem of computing the points that can be in contact with a robotic welding head when the welding head must have its tip at a weld site and must be within a specified range of the optimal weld angle. We show that these points form a solid of revolution. If the welding head is represented as a polygon with n vertices, we give an $O(n^2 \log n)$-time algorithm to compute a two-dimensional representation of the entire solid; we also give an $O(n \log n)$-time algorithm to compute a two-dimensional representation of the outer surface of the solid. This representation permits the use of existing computer-aided design tools for collision avoidance.

1 Introduction

One of the enjoyable aspects of computational geometry is how an understanding of the geometry of a problem can lead to simple and elegant solutions. Our case in point identifies a representation for the *reachable region* of a robotic welding head, the set of all points that can be in contact with the head. While the reachable region is a three-dimensional volume, characteristics of the head's motions permit a compact two-dimensional representation.

We have a robotic welding head, which has a welding tip and an axis ℓ that passes through the tip. The overall problem is to track a weld seam with the welding tip so that the axis ℓ is always within some angle δ of an optimal weld angle that has been specified by some expert welder.

We attach a coordinate frame to the seam with the origin o at a point *weld site* so that the welding head axis coincides with the z-axis at the optimal angle for welding site o. As figure 1 illustrates, the welding head has two types of motion when the tip is at o: It can rotate freely about its axis ℓ, and ℓ can tilt within an angle of δ from the z-axis. A subproblem that arises is to determine the (presumably few) features of the (presumably large) workspace that must be avoided in tracking the seam. Given a representation of the *reachable region* of the head, this can by computed by commercial CAD packages using swept-volume and intersection operations.

In this paper we abstract the welding head problem to computing the reachable region of a connected polygon P with n vertices. We prove in section 2 that the reachable region is a solid of revolution. Thus, rather than explicitly constructing the three-dimensional region, we create a two-dimensional cross-section, bounded by line segments and circular arcs. Our algorithm, described in section 3, runs in $O(n^2 \log n)$ time; the cross-section computed may have quadratic

*This work has been partially supported by an NSERC Research Grant and an NSERC Post-Graduate Scholarship. Portions of the second author's research were carried out while visiting the Max-Planck-Institut für Informatik.

complexity due to ringlike holes inside the solid. The outer surface of the reachable region has at most $O(n)$ complexity; we can compute its cross-section in $O(n \log n)$ time.

2 Geometry of the Welder Problem

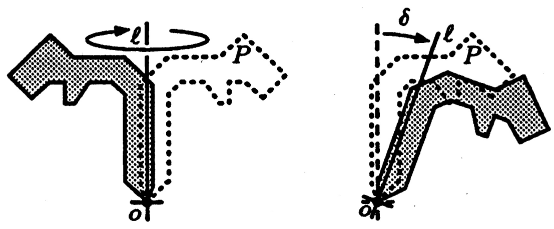

Figure 1: P rotates about ℓ while ℓ tilts about the origin o.

We abstract the problem as follows.

The welder problem: Let P be a connected polygon with n vertices in a plane π, let ℓ be a line in π, and let o be a point on ℓ. Find a two-dimensional representation for the volume swept by the interior of P as P is rotated freely about ℓ and ℓ is tilted by at most δ radians about the point o (see figure 1.)

As our first task, we show that the welder problem defines a solid of revolution. A two-dimensional cross-section completely describes the entire volume, so the welder problem is feasible. We then characterise the cross-section as our second task, which leads us toward an efficient computation algorithm.

Lemma 1 *The volume swept by the interior of a plane connected polygon P as P rotates about a co-planar line ℓ and ℓ tilts by angle at most δ is a solid of revolution.*

Proof: We prove that every point within the swept volume lies on a circle that is centred on ℓ, is perpendicular to ℓ, and lies completely within the volume. Since all the motions involve rotations about an origin o, every point simply moves on the surface of a sphere, so we show that any arbitrary point inside the volume can move to a plane containing ℓ that makes any angle relative to π with a motion perpendicular to ℓ, and remain inside the swept volume.

Let p be an arbitrary point inside the swept volume. The point p corresponds to some point inside the polygon P after P and ℓ are tilted by at most δ radians and then P is arbitrarily rotated by ψ radians about the tilted ℓ. The tilt motion of ℓ divides into two independent components: an angle ϕ with respect to the untilted ℓ, and an angle θ between the plane π and the plane containing the untilted and the tilted copies of ℓ. Each of ψ, ϕ and θ are positive angles; angles θ and ψ are unconstrained while angle ϕ is no greater than δ.

If we use a rectangular coordinate system centred at o, with the z-axis along the line ℓ, and if the initial position of the point p relative to this coordinate system, before the ψ, ϕ, and θ rotations, was the point (a, b, c), then the projection of p onto the line ℓ has coordinates $(0, 0, (a \sin\psi + b \cos\psi) \sin\phi + c \cos\phi)$. Since the projection along ℓ is independent of the rotation angle θ, any motion of p caused by varying θ, while ψ and ϕ are fixed, moves p in a plane perpendicular to ℓ. The welder problem does not constrain θ, so the point p can traverse a circle centred about ℓ and perpendicular to ℓ without violating any of the volume constraints. ∎

Lemma 1 proves the existence of a two-dimensional representation for the volume in the welder problem. A cross-section of the volume, which contains the line ℓ, completely describes the volume.

A direct approach for obtaining the two-dimensional cross-section computes the entire volume and then extracts the desired profile.

Each possible polygon motion is treated separately; first, we revolve the polygon P about the line ℓ to produce a volume V relative to ℓ, then we take the union of the volume V as the reference line ℓ is positioned at every possible tilt angle and tilted in every possible direction. The resulting union contains exactly all possible positions for the interior of P through all motion combinations. Since we know the reachable region is a solid of revolution and ℓ is its axis of rotation, we can compute the cross-section through ℓ and obtain a two-dimensional answer.

The welder problem's geometry allows us to compute the two-dimensional cross-section without calculating in the dimension of the reachable region itself. Lemma 2 proves that any point of the reachable region in the plane π, attainable by P in the welder problem, is also attainable by a single and permissible tilt of P and ℓ within the plane π alone. Let P_θ denote the polygon P tilted about the origin by the angle θ and let v_θ denote the image of vertex $v \in P$ on P_θ. Then lemma 2 and the observation that any tilt between $P_{+\delta}$ and $P_{-\delta}$ is a valid motion for the polygon describe the two-dimensional cross-section of the volume as the union of $P' = \bigcup_{\theta \in [-\delta, +\delta]} P_\theta$ and its reflection in the line ℓ.

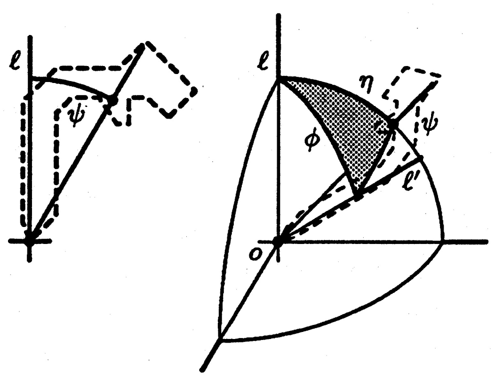

Figure 2: Triangle on the surface of a sphere with subtending angles ϕ, η, and ψ.

Lemma 2 *Let polygon P in plane π be defined as in the welder problem. If $p \in P$ maps to the point $p' \in \pi$ after a tilt of ℓ by ϕ radians in any direction followed by rotation of P about the tilted ℓ then point p can be mapped to p' by a tilt of ℓ in π of ϕ or fewer radians.*

Proof: We show that the angle between p and p' is less than ϕ in absolute value by using the triangle inequality on the surface of the sphere containing p and p'.

First, we can assume that points p and p' lie in the same half-plane of π relative to ℓ since the polygon P can be rotated about ℓ to satisfy this condition before we start.

Let ψ be the measure of the angle in π between ℓ and the line from o to p. Let η be the measure of the angle in π between ℓ and the line from the origin o to p' (see figure 2).

The angle between points p and p' in the plane π has measure $|\eta - \psi|$. The points p and p' are equidistant from the centre of rotation and therefore lie on the surface of a common sphere centred at o. The triangle on the surface of this sphere, with vertices p, p', and the intersection of ℓ and the sphere, has edge lengths proportional to η, ψ, and ϕ as illustrated in figure 2. The triangle inequality, when applied twice to this triangle, yields

$$\begin{aligned} \psi &\leq \eta + \phi \\ \eta &\leq \psi + \phi \end{aligned}$$

We rewrite the inequalities as

$$\begin{aligned} -\phi &\leq \eta - \psi \\ \eta - \psi &\leq \phi \end{aligned}$$

to imply $|\eta - \psi| \leq \phi$. ∎

The geometry of the welder problem reduces the two-dimensional representation problem to creating the union of the polygon P' and its reflection in ℓ.

3 Computing the polygon generating the reachable region for the welding head

In this section we look at the algorithmic problems of computing the two-dimensional representations of the reachable region for a welding head that is given as a polygon P with n vertices. We compute the polygon generating the reachable region in $O(n^2 \log n)$ time and $O(n^2)$ space. The space is optimal in the worst case—the polygon generating the reachable region can have quadratic complexity because the reachable region can have many ring-shaped cavities. For collision detection, however, these cavities are not interesting. Thus, we also compute the polygon generating the outer boundary of the reachable region in $O(n \log n)$ time and linear space.

We begin by describing how to compute the area swept by a connected plane polygon Q, with m vertices, as Q tilts in the plane by angles in the range $[-\delta, +\delta]$. That is, how to compute $Q' = \bigcup_{\theta \in [-\delta,+\delta]} Q_\theta$. By introducing at most m vertices, we can ensure that each edge of Q has a vertex as the closest point to the origin; along an edge the distance from the origin increases or decreases monotonically. Because distance from the origin is monotone along an edge of Q, we can say unambiguously that an edge e is clockwise (cw) or counterclockwise (ccw) depending on whether e as viewed from the origin is cw or ccw of the interior of Q.

Lemma 3 *For any connected plane polygon Q, the boundary of $Q' = \bigcup_{\theta \in [-\delta,+\delta]} Q_\theta$ consists of cw edges from $Q_{-\delta}$, ccw edges from $Q_{+\delta}$, and arcs of circles centred at the origin generated by vertices of Q that are local minima or maxima with respect to distance from the origin to the boundary of Q. At most one portion of each arc appears on Q'.*

Proof: Lemma 2 implies that the boundary of Q' consists of straight edges and circular arcs. Since tilting about the origin is distance preserving, our classification of edges as cw and ccw relative to Q applies to the edges of Q'.

Suppose e' is a cw edge of Q'. The angle between e' and its corresponding edge e in Q is δ, so e' belongs to either $Q_{-\delta}$ or $Q_{+\delta}$ and no other tilted polygon. Since all the points of $Q_{+\delta}$ lie ccw to Q and, by definition, e' lies cw to Q, e' must belong to $Q_{-\delta}$. A parallel argument shows that ever ccw edge of Q' comes from $Q_{+\delta}$.

Let v be a vertex of Q. If v is not a local minimum or maximum with respect to distance from the origin, then tilting Q about the origin sweeps a circular band about the path traced by v, hiding v's arc from the boundary of Q'. Otherwise, let γ be the arc from $v_{-\delta}$ to $v_{+\delta}$. At most one portion of γ can appear on the boundary of Q': If γ intersects a cw segment of Q' then it has encountered a segment of $Q_{-\delta}$; because γ subtends an angle of 2δ, it cannot reappear on Q' ccw of this intersection. A similar analysis says that γ cannot appear cw of its first intersection with $Q_{+\delta}$. ∎

The observations of lemma 3 suggest an algorithm to compute the arcs of Q'. We sweep the polygon Q with an expanding circle centred at the origin. We maintain ordered lists of the cw segments of $Q_{-\delta}$ and of the ccw segments of $Q_{+\delta}$ that intersect the circle. When the sweep encounters a vertex $v \in Q$, we update the appropriate list(s). If v is a local minimum or maximum of distance from the origin (equivalently, if we must update both lists) then we locate the ccw neighbor of $v_{-\delta}$ among the cw segments, calling $v_{+\delta}$ the ccw neighbor if there is none. Similarly, we locate the cw neighbor of $v_{+\delta}$ among the ccw segments, calling $v_{-\delta}$ the neighbor if there is none. If there is a portion of the arc clockwise from the ccw neighbor to the cw neighbor, then we introduce new vertices where the arc intersects the segments and set pointers between them.

Now, we must put the cw segments of $Q_{-\delta}$ and the ccw segments of $Q_{+\delta}$ together with the arcs. Perhaps the simplest way would be to adapt a line-sweep algorithm for computing line segment intersections, such as Bently-Ottman [1], to a circle-sweep. This, however, could have a running time that is quadratic in m because of internal intersections. Instead, we preprocess the components of the exteriors of $Q_{-\delta}$ and $Q_{+\delta}$ for ray shooting [3, 4, 2]. We also mark vertices that will appear on the final output: perform a circle-sweep of Q and mark vertices of cw segments (including those that we have introduced) where the angle to the cw neighbor is greater than 2δ.

From a marked vertex we walk around a boundary component of Q' by alternately walking on $Q_{-\delta}$ and $Q_{+\delta}$ as follows: Unmark the current vertex. If the next edge (keeping the interior of Q' to the left) is an arc, then follow the pointer to the vertex on the other polygon. If the next edge is a segment e, then shoot along e in the other polygon's ray shooting structure to determine if e hits the other polygon. If e does not, then advance to the other endpoint of e. Otherwise, introduce the intersection point and make it the current vertex on the other polygon. Repeat until there are no marked vertices.

An intersection found by ray shooting is the intersection of a cw and a ccw segment—it is, therefore, a local minimum or maximum of its boundary component. Because every boundary component of Q' has at least three vertices, every boundary component has at least one marked vertex and is output by the algorithm.

The circle-sweeps handle a linear number of events; they can be implemented to run in $O(m \log m)$ time using balanced trees. Preprocessing and ray-shooting from the $O(m)$ vertices can be performed in the same time. We summarize with the following theorem.

Theorem 4 *Given a connected plane polygon Q with m vertices, we can compute the boundary of $Q' = \bigcup_{\theta \in [-\delta, +\delta]} Q_\theta$ in $O(m \log m)$ time and $O(m)$ space.*

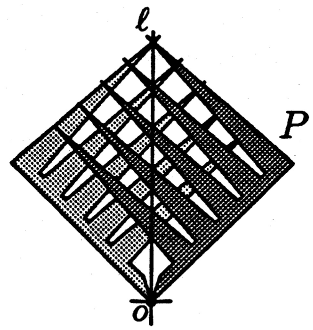

Figure 3: A polygon and its reflection can have quadratic complexity

Section 2 characterises the polygon generating the reachable region for a welding head represented as an n-gon P. We can compute it by computing the polygon Q that is the union of P and P's reflection across ℓ and applying theorem 4.

Corollary 5 *One can compute the polygon generating the reachable region for a welding head P in $O(n^2 \log n)$ time and $O(n^2)$ space.*

Proof: Compute the union of P and P's reflection by a line-sweep algorithm [1]. This takes $O(n^2 \log n)$ time and results in a connected polygon Q with $O(n^2)$ vertices. Apply the algorithm of theorem 4 to finish the computation. ∎

As figure 3 shows, the description of the polygon generating the reachable region may have quadratic complexity, even with little or no tilting. In fact, it is the reflection and not the tilting that causes the increase in complexity—Theorem 4 implies that the tilt of P itself has $O(n)$ complexity. (This can also be seen by noticing that the n tilted segments form a family of *pseudodisks*—sets whose boundaries have two intersections—and applying a result of Kedem et al. [5].)

The reachable region or solid of revolution generated by such a polygon contains many ringlike holes, which might as well be filled in if the goal is to perform collision detection with obstacles that are not freely floating in space. It is sufficient to compute the *contour*, the polygon generating the outer boundary of the reachable region, which has only linear complexity.

Corollary 6 *One can compute the contour of P, the polygon generating the outer boundary of the reachable region, in $O(n \log n)$ time and linear space.*

Proof: First, compute the contour of the union of P and P's reflection about the line ℓ. For this we process the exteriors of P and its reflection for ray-shooting. Then we walk around their union starting from their rightmost point in the manner of the algorithm for theorem 4. In $O(n \log n)$ time, this produces a polygon Q with $O(n)$ complexity. Applying theorem 4 (and generating only the outer boundary component) completes the computation. ∎

4 Future Work

Our work abstracts a welding head as its polygonal profile. A true representation of the reachable region for the head must also account for depth of the head, information available through head descriptions as a union of polyhedra (common in computer graphics) or as front, side, and plan views of the head (from traditional drafting). With depth information and the same motion constraints, the reachable region remains a solid of revolution and a two-dimensional representation exists. However, a new characterisation of the reachable region's outer surface as well as new algorithms to outline the outer surface must be developed to include the depth factor.

Restricted forms of the welder problem also provide a space for further analysis. If the polygon cannot be rotated completely about the line ℓ, if the polygon cannot be tilted in an arbitrary direction, or if the polygon cannot be tilted in each direction uniformly then we no longer obtain a solid of revolution as the reachable space. The loss of symmetry leaves the question of whether or not a two-dimensional representation exists for the reachable space and, if it exists, how complex that representation is and how fast can we compute the representation?

5 Acknowledgement

We thank Dan Li for bringing this problem to our attention and Stephan Schirra for discussions on the contour.

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