Optimal Floodlight Illumination of Stages

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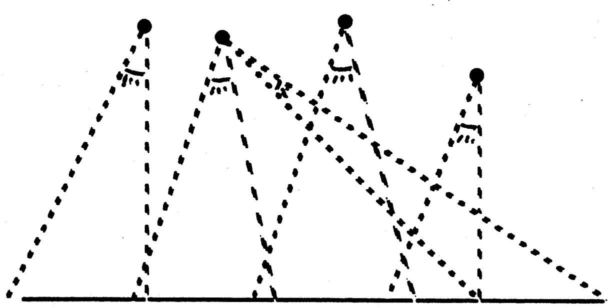
1. Introduction

Illumination problems of several kinds have been studied intensely in recent years [6]. One of the first results in this area is that of Chvatal [2] which states that any simple polygon with n vertices can be guarded (in the context of this paper *illuminated*) using at most $\lfloor\frac{n}{3}\rfloor$ lamps. It is also known that any family of n disjoint compact convex sets can be illuminated using at most 4n-7 lamps [5]. Numerous variations of these problems have been studied in the literature; see [1,2,3,4,5,6].

Normally it has been assumed that the light sources used emit light in all directions. In this paper we study a line illumination problem in which our light sources have a restricted angle of illumination, just the way a floodlight works. Thus for the rest of this paper, a floodlight is a source of light located at a point p of the plane with an angle of illumination α.



A set of five floodlights that illuminates L

Figure 1

Floodlights were first introduced in [1] where the following problem, called "The Stage Illumination Problem" is studied:

Given a stage (represented by a line segment L) a set $F=\{f_1,...,f_n\}$ of floodlights located at some predetermined locations (a set of points on the plane all on the same side of L), is it possible to rotate the floodlights around their positions in such a way that the stage is completely illuminated? (See Figure 1.)

Given a set $F = \{f_1,...,f_n\}$ of floodlights each with angle α_i, i=1,...,n, we can associate to F an angular cost :

$$\alpha(F) = \sum_{i=1}^{n} \alpha_i.$$

In this paper we study the following stage illumination problem: Given a stage, represented by a line segment S and a set $P=\{p_1,...,p_n\}$ of n points, determine a set of floodlights F that illuminates S such that the angular cost of F is minimized and each floodlight $f_i \in F$ is located at some point $p_j \in P$. From now on we shall refer to this problem as the "Floodlight Illumination Problem of S from P". In this problem we allow for more than one floodlight to be placed at any given point. We give an optimal O(n log n) time algorithm to solve this problem.

2. Optimal Floodlight Illumination of the Real Line

To solve our floodlight illumination problem for line segments, we first solve the problem of optimal floodlight illumination of the real line L, namely: Given a set of points $P = \{p_1,...,p_n\}$ on the plane, all on the same side of a line L, find an optimal floodlight illumination of L from P. It is straightforward to show that a solution to the line segment illumination problem can be obtained by restricting the solution to the real line L to S. Without loss of generality, let us assume that the distances between L and all of the elements of P are different, that L is the x-axis and that all of the points of P have y-coordinate greater than 0. From now on, we say that a point x of L is illuminated from a point p_i of P if x is illuminated by a floodlight of F placed at P_i.

In the rest of this paper, a disk D_i will refer to a circle C_i together with the set of all points contained within C_i.

We consider first the floodlight illumination problem of the real line L from two different points p_i and p_j. Assume without loss of generality that p_i is closer to the real line than p_j. (See Figure 2.)

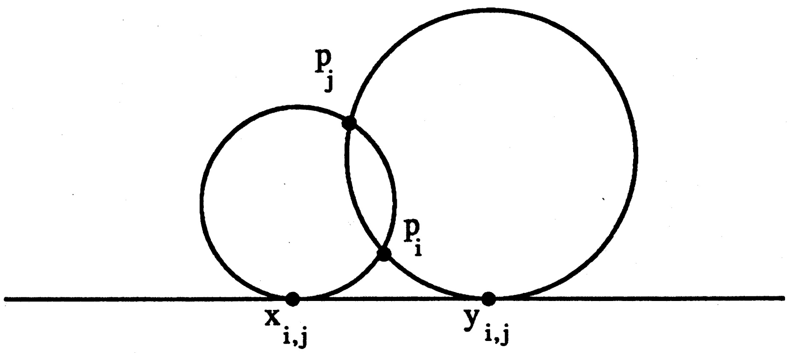


Figure 2

Consider the two circles that contain p_i, p_j and are tangent to L. Let us denote the tangency points of these circles and L by $x_{i,j}$ and $y_{i,j}$; $x_{i,j}$ always being to the left of $y_{i,j}$. The following lemma given without proof solves the floodlight illumination problem of L from $\{p_i, p_j\}$

Lemma 1. In the optimal floodlight illumination of the real line L from $\{p_i, p_j\}$ all points in the interval $[x_{i,j}, y_{i,j}]$ are illuminated from p_j and all points in the intervals $(-\infty, x_{i,j}]$ and $[y_{i,j}, \infty)$ are illuminated from p_i.

Lemma 2. Let $P=\{p_1,\ldots, p_n\}$ be a set of n points and p_j a point in the interior of the convex hull of P. Then in an optimal floodlight illumination of L with a set of floodlights F, no element of F is located at p_i.

Proof: Suppose that p_i is an interior point of the convex hull of P, and that there is an optimal illumination of L in which a floodlight L_i of F placed at p_i illuminates an interval, say [x,y], of L. Consider the smallest disk D containing x, y and all of the elements of P. (See Figure 3.)

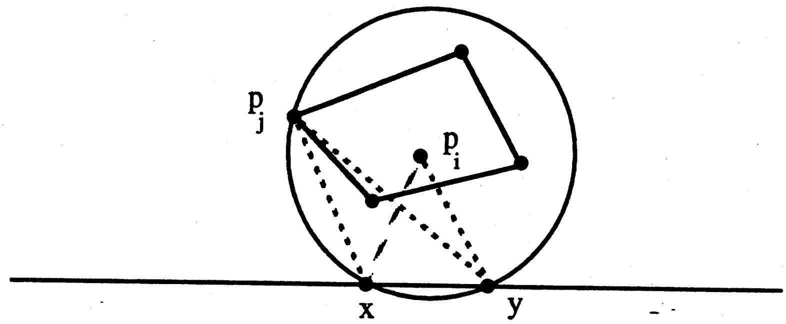


Figure 3

Let p_j be a point of P located in the boundary of D. Since p_i is in the interior of the convex hull of P, $p_i \neq p_j$; moreover p_i belongs to the interior of C. Therefore the angle $\angle x,p_i,y$ is greater than angle $\angle x,p_j,y$. Thus we could substitute the floodlight at p_i that illuminates the interval $[x,y]$ by a smaller one placed at p_j that illuminates the same interval. This contradicts our assumption on the optimality of F.

QED.

For any point x in L and a point p not in L let $C(x,p)$ $(D(x,p))$ be the circle (disk resp.) tangent to L at x and containing the point p. The following result is an easy consequence of Lemmas 1 and 2:

Lemma 3: Consider an optimal floodlight illumination of L with respect to P, and a point x of L. Then if x is illuminated by a floodlight of F placed at a point p_i of P, the disk $D(x, p_i)$ contains all of the elements of P.

Let us assume without loss of generality that p_1 is the element of P closest to L. We can prove:

Lemma 4: Let x be the leftmost point of the set $\{x_{1,j}; j \neq 1\}$ and y the leftmost point of the set $\{y_{i,j}; j \neq 1\}$. Then all the points to the left of x and those to the right of y are illuminated by floodlights placed at p_i.

Proof: It is easy to see that for any point q to the left of x or to the right of y the disk $D(q,p_1)$ contains all the elements of P. The result now follows from Lemma 3.

QED.

For example, in Figure 4, all the points of L to the left of $x = x_{1,5}$ and those to the right of $y = y_{1,2}$ will be illuminated from p_1.

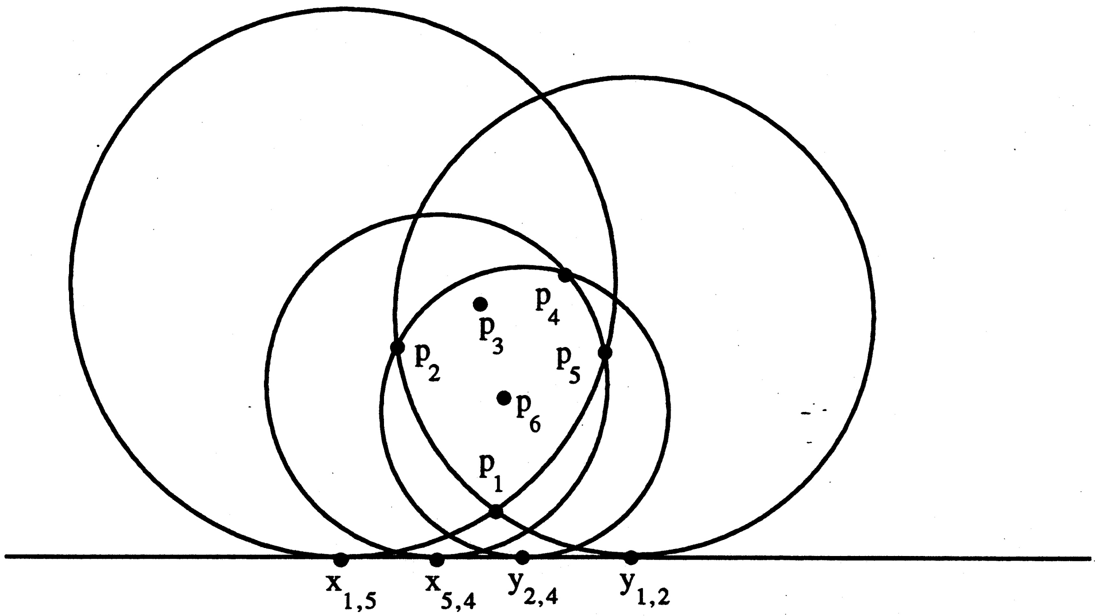


Figure 4

Using Lemma 3, we now proceed to develop an algorithm to solve our floodlight illumination problem:

Algorithm FLIP

Input: A set $p = \{p_1,...,p_n\}$ of n points on the plane with positive y-coordinates, and the real line L.

Output: A partitioning of the lines into a sequence of at most $n+1$ intervals, each assigned to one point of P from where that interval is to be illuminated.

Step 1: Calculate the convex hull of P. Relabel the vertices of the convex hull of P in the clockwise direction by $\{p_1,...,p_k\}$ where p_1 is the point of P closest to L and k is the number of vertices of the convex hull of P.

Step 2: Determine the point $y = y_{1,i}$ = rightmost point in $\{y_{1,j}; j=2,\ldots,n\}$. Illuminate all of the points in the interval $[y,\infty)$ from p_1.

While $i \neq 1$ **do:**

a) Find the smallest index $j>i$ (or take $j=1$ if no such j exists) such that the disk D defined by the circle tangent to L containing p_i and p_j contains all the elements of P. Let x be the point in which C is tangent to L.

b) Illuminate the interval $[x,y]$ from p_i.

c) $i \leftarrow j$, $y \leftarrow x$

EndWhile

Step 4: Illuminate the interval $(-\infty,y]$ from p_1. **Stop**

For example in Figure 4, y initially takes the value $y_{1,2}$, and i the value 2. In the next iteration, y changes to $y_{2,4}$ and i to 4. Notice that even though p_3 is a vertex in the convex hull of $P=\{p_1,\ldots, p_6\}$, no interval is illuminated by p_3. This happens because the circle tangent to L through p_2 and p_3 does not contain p_4, and in the execution of our While loop for i=2, j skips the value 3. The subsequent values for y are $x_{5,4}$ and $x_{1,5}$ and the values for i are 5 and 1 respectively. Thus all the points in the intervals $(-\infty, x_{1,5}]$ and $[y_{1,2}, \infty)$ are illuminated from p_1, and $[y_{2,4}, y_{1,2}]$, $[x_{5,4}, y_{2,4}]$ and $[x_{1,5}, x_{5,4}]$ are illuminated from p_2, p_4 and p_5 respectively.

3. Correctness and Complexity Analysis of Algorithm FLIP

The correctness of our algorithm follows from Lemma 3 and the observation that when executing the While part of our algorithm for a given value of i, for any point q in the interior of the interval $[x,y]$ defined in our loop, the circle tangent to L at q containing p_i contains all the points of P. Referring again to Figure 4, for any point q, say between $y_{2,4}$ and $y_{1,2}$, the circle tangent to L at q containing p_2 contains all the elements of $P=\{p_1,\ldots,p_6\}$.

For the complexity analysis, Step 1 takes O(n log n), step 2 take together O(n). It remains only to show that all the iterations of our While loop can be done in O(n log n) time. We notice first that using Voronoi diagrams, we can test in logarithmic time if a circle C tangent to L containing two elements p_i and p_j of P encloses all the elements of P. We simply test if the farthest sites from the centre of C are precisely p_i and p_j [7]. Thus each execution of Step 3(a) can be performed in O(log n) time. We also notice that the number of times Step 3(a) is executed is exactly n, thus obtaining our result. To prove that our algorithm is optimal, we can show that sorting can be solved by FLIP. Given a set of numbers $\{x_1,\ldots,x_n\}$ all we have to do is to map them into a set of points $P=\{(x_1,y_1),\ldots,(x_n,y_n)\}$ such that all he elements of P lie on the top right part of C. (See Figure 5.) It is esy to see that the order in which the intervals of illumination in which L is subdivided by FLIP correspond to the reverse sorted order of {x1,...,n}.

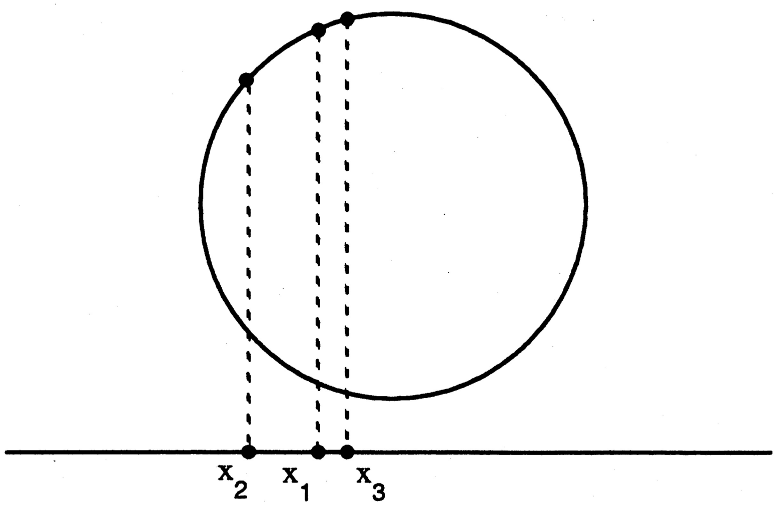


Figure 5

Summarizing, we have:

Theorem: The Floodlight Illumination Problem of L from P can be solved optimally in O(n log n) time.

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