Reconstructing Polygons from X-Rays (Extended Abstract)

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1 Introduction

CAT scanners and other tomographic imaging systems reconstruct cross-sectional images of internal structures, providing an important tool for diagnosing tumors and other medical problems. Tomographic scanners estimate line integrals by sending an energy pulse of some type through the object and then measuring how much energy is absorbed.

The fundamental mathematical result upon which tomography is based is the Radon inversion theorem [17], which presents how to unambiguously reconstruct any well-behaved function $f(x, y)$ in two variables from the complete set of line integrals through f. In general, the complete set is necessary for reconstruction. However, in this paper we shall show that if $f(x, y)$ is restricted to be a very general class of geometric objects, then a finite collection of x-ray directions are sufficient for determination. For more extensive surveys on tomography, see [12, 13, 19, 23].

Edelsbrunner and Skiena [3] first considered the problem of interactively reconstructing convex polygons from carefully chosen line integrals or *x-ray probes*, and showed that $5n + 19$ x-ray probes are sufficient to reconstruct an arbitrary convex n-gon P given only a point within P to start with. That $3n - 3$ x-ray probes are necessary follows from Lindenbaum and Bruckstein's [15] lower bound for determination where two opposing finger probes on a line l count as a single probe, since $P \cap l$ can be computed from the two contact points. Geometric probing problems were first introduced by Cole and Yap [2], and have since inspired a significant literature, as surveyed in [20, 22].

The techniques used by Edelsbrunner and Skiena for convex polygons cannot extend to more general polygons, as sufficiently small notches cut into the face of P will be undetectable to any finite number of line integrals. However, actual tomographic sensing devices do not measure isolated line integrals, but instead simultaneously measure all line integrals with a particular slope (parallel probes) or through a particular point (origin probes). In [20], it is shown that three parallel probes or two origin probes are necessary and sufficient to determine convex n-gons. Recently, this result was generalized by Gardner and Gritzmann [8], who also show that $\lfloor d/(d-k) \rfloor + 1$ k-dimensional x-ray probes are necessary and sufficient to verify a convex polytope in E^d. Narasimhan [16]

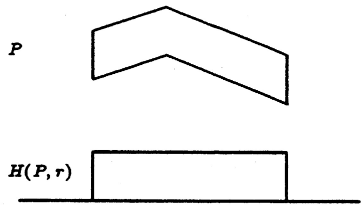

Figure 1: Invisible vertices in simple polygons.

shows that three parallel probes suffice to determine any n-gon, provided that none of the probing directions correspond to directions of the polygon. However, it is impossible to detect whether such a degeneracy has occurred without making additional probes.

In this paper, we generalize the results for x-ray probes to simple polygons. Boissonnat and Yvinec [1] have considered the problem of finger probing non-convex polyhedra. For x-ray probes, the loss of convexity complicates matters considerably, due to the potential existence of invisible vertices in the histogram resulting from a parallel probe, as shown in Figure 1.

This problem of invisible vertices has lead to the study of k-projections, discussed in [21]. A k-projection of a set of n points is an orthogonal projection that yields at most k point images. The study of k-projections provides insight into how many invisible vertices can possibly remain after m parallel probes.

The problem of reconstructing convex sets from a *predetermined* set of x-ray parallel or origin probes was posed by P. C. Hammer [11] in 1963, and has since generated a substantial literature [4, 5, 6, 9, 10, 18, 24] based on integral geometry. Recently, Gardner [7] proved that star-shaped polygons cannot be reconstructed from a constant number of parallel x-ray probes from predetermined directions. In this paper, we prove a logarithmic lower bound for the much stronger case of interactive reconstruction, although our polygons are not star-shaped.

2 Preliminaries

Let P be polygon of known and uniform density. A parallel x-ray probe through a polygon P with direction r computes a histogram $H(P,r)$, which records the line integral $l(P)$ for every line l of the form $l : rx + b$, where $-\infty < b < \infty$. Observe that if P is a polygon with n vertices, $H(P,r)$ is a polygon containing at most n vertices, as shown in Figure 1.

The existence of invisible vertices ensures that one parallel x-ray probe is insufficient for reconstructing the edges of P. However, the following lemma shows that two well-placed x-ray probes determine the supporting lines of a pair of edges of P.

Lemma 1 *Let l_1 and l_2 be two non-parallel lines and O be a known vertex of polygon P. Let w be the smallest wedge between two half-lines emerging from O and intersecting l_1 and l_2. Then w is uniquely determined by the lengths of $l_1 \cap w$ and $l_2 \cap w$.*

Proof: Without loss of generality, assume that l_1 is a vertical line and that O is to the left of l_1 and above l_2. Consider the situation in Figure 2. For $i = 1, 2$, let x_i be the distance between O and l_i. Let ϕ be the size of the positive angle from l_2 to l_1. Let θ be the angle between a horizontal half-line emerging from O to the right and a half-line emerging from O which intersects l_1 and l_2.

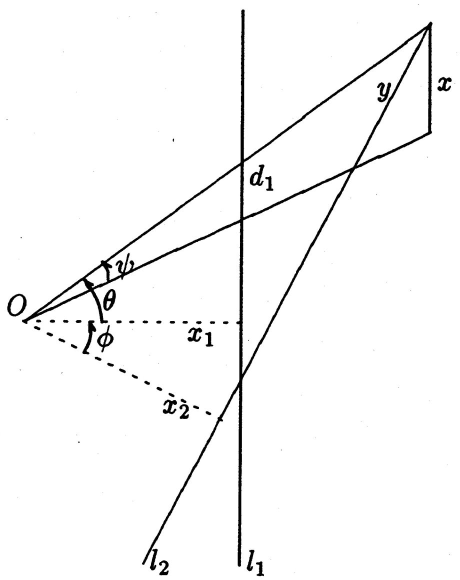

Figure 2: Two non-parallel line integrals determine a wedge.

So $-\pi/2 < \theta < \pi/2 - \phi$. For each θ, we can construct a wedge w, such that length $l_1 \cap w = d_1$ and the upper half-line of the wedge has angle θ. Let ψ be the positive angle between the lower and the upper half-line of w. If we define $y = l_2 \cap w$ then we can express y as a function of θ:

$$y(\theta) = \frac{x}{\cos\phi - \sin\phi \tan(\theta - \psi)}$$

where

$$x = d_1 \cdot \frac{x_2}{\cos(\theta + \phi)} \cdot \frac{\cos(\theta)}{x_1}$$

The numerator x is an increasing function of θ on the interval $(-\pi/2, \pi/2 - \phi)$ and the denominator of $y(\theta)$ is a positive decreasing function of θ on $(-\pi/2, \pi/2 - \phi)$. So $y(\theta)$ is an increasing function. Since it maps the interval $(-\pi/2, \pi/2 - \phi)$ onto the interval $(0, \infty)$, there is a unique θ with $y(\theta) = d$ for any positive value of d. In the full paper, we derive a closed form for θ. ∎

Lemma 1 provides a tool to reconstruct edges once the incident vertex and an enclosing hull are known. Despite the complications of invisible vertices, there exists a class of vertices which can be easily identified.

Lemma 2 *The convex hull of any polygon P with h hull vertices can be determined in $3h - 2$ parallel x-ray probes.*

Proof: By ignoring all but the shadow cast by each probe, we obtain two supporting lines on the outer boundary of P, each parallel to the direction of probing. Thus each parallel x-ray probe can be interpreted as providing exactly the same information as a silhouette probe sent perpendicular to the direction of probing. Using Li's determination algorithm for silhouette probes [14], the convex hull of P can be determined in $3h - 2$ probes. ∎

3 Interactive Determination Strategies

Invisible vertices complicate determination for simple polygons. In this section, we provide a determination strategy for a slightly less general class of geometric objects, namely polygons whose vertices are in general position. Further, we determine arbitrary simple polygons when an upper bound on the number of vertices is given.

Theorem 1 *Let P be a simple polygon where no three vertices are collinear, containing an unknown number of vertices n. Then $n + h + 2$ parallel x-ray probes are sufficient to determine P, where h is the number of vertices on the convex hull of P.*

Proof Sketch: As described in Lemma 2, every parallel probe defines two supporting lines for P. It can be shown that after at most $h + 1$ probes a convex hull vertex v of P can be found. After one more probe, the rays incident on v can be found. Thus after $h + 2$ probes, we can determine a hull vertex and its incident rays.

Now suppose that we know the direction of an edge $(v_{i-1}v_i)$, the side of this edge that is on the interior of P and the direction r of the edge (v_iv_{i+1}). The histogram $H(P,r)$ allows us to find the next vertex. Because no three vertices are collinear, there will be a vertical edge e on $H(P,r)$ at v_i, and the length l of this edge will be exactly the length of the edge (v_iv_{i+1}). If $H(P,r)$ immediately to the left of edge e is larger than $H(P,r)$ immediately to the right of e, then v_{i+2} is on the same side of $(v_{i-1}v_i)$ as the interior of P. Otherwise v_{i+2} is on the opposite side. Therefore, $H(P,r)$ uniquely determines v_{i+1} and the side of the edge (v_iv_{i+1}) that is inside P. The change in slope between the edges before and after e on $H(P,r)$ is equal to the change in directions between the edges $(v_{i-1}v_i)$ and $(v_{i+1}v_{i+2})$. So we can compute the direction of the line supporting the edge $(v_{i+1}v_{i+2})$.

Therefore, after finding a convex hull vertex plus its incident rays in $h + 2$ probes, we can compute the remainder of P in n probes, giving the result. ∎

Theorem 2 *Let P be an arbitrary simple polygon, and n' be an upper bound on the number of vertices of P. Then $2n' + 2$ probes are sufficient to determine P.*

Proof: The key to this strategy is the observation that all the vertices of P can be determined with any set of $2n'$ distinct probes. Each probe defines a set of up to n' parallel lines along which at least one vertex of P must lie. Construct the arrangement of these $2n'$ sets of lines. It can be shown that the vertices of P are exactly the points in this arrangement of degree at least $n' + 1$.

Once the vertices of P have been established, identify a direction θ where no two vertices are collinear. Drawing lines with this slope through each vertex partitions P into slabs, where no slab contains a vertex. Now send two probes at directions r_1 and r_2 close enough to θ such that for each direction there are lines l_1 and l_2 along r_1 and r_2 through each slab which intersect P completely within the slab.

By using Lemma 1 it can be shown that these two probes are sufficient to determine the edges of P, so The total number of probes required to determine P is $2n' + 2$. ∎

A frustrating aspect of reconstruction via parallel x-rays is that with probability one, after three random parallel x-ray probes, all the vertices of P will be determined, since only $n(n-1)/2$ directions are defined by the vertices of the polygon. Once these are determined, two extra probes aimed within a slab suffice to complete reconstruction, by the sweepline argument of Theorem 2. Similarly, 5 x-rays are sufficient to determine a simple ploygon P if its vertices have coordinates on a rational grid.

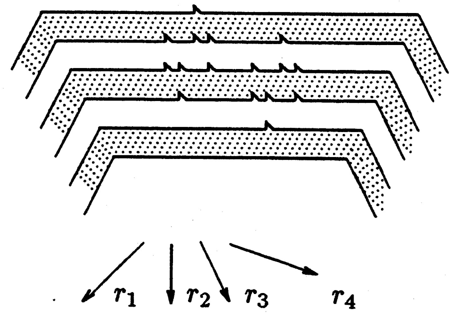

Figure 3: Mutually complementary notches along four directions.

Gardner [7] proves that star-shaped polygons cannot be reconstructed from a constant number of parallel x-ray probes from predetermined directions. Here we prove a higher lower bound for the more challenging interactive case, although our proof does not hold for star-shaped polygons.

Theorem 3 *Let P be a simple polygon with n vertices. Any interactive strategy which reconstructs P using parallel x-ray probes requires at least $\lfloor \lg n - 2 \rfloor$ probes for $n \geq 16$.*

Proof Sketch: Consider a polygon P, which resembles one half of a regular $2(k+1)$-gon with $2\lceil (k+1)/2 \rceil$ edges along each side, connected to form a polygon. Suppose the histograms $H(P, r_i)$ for $1 \leq i \leq t-1$ of polygon P have been obtained.

For any particular direction r, we can construct a polygon P', such that $H(P, r) = H(P', r)$ by adding a pair of complementary triangular notches, as shown in Figure 3. In general, for a given set of directions r_i, $1 \leq i \leq t$, a set of 2^t mutually complementary notches can be constructed such that $H(P, r_i) = H(P', r_i)$ for $1 \leq i \leq t$.

The sides of P have angles $\theta_i = i\pi/(k+1)$ for $0 \leq i \leq k$. The set of notches can be added to a particular side of P with angle θ if none of the directions r_i, $1 \leq i \leq t$ has an angle within ϵ of θ for some positive value for ϵ. If the edges of P are sufficiently long, ϵ will be small enough for the intervals $[\theta_i - \epsilon, \theta_i + \epsilon]$ not to overlap for $0 \leq i \leq k$. In this case, for any set of k directions there is always one side of P to safely hide the notches. ∎

4 Polygon Verification Strategies

The results in Section 3 require trusting our adversary to confirm that the polygon P contains either no three vertices which are collinear or provide us with an upper bound n' on the number of vertices of P. A finite determination strategy for the general case would follow if we could develop a finite strategy to verify whether a conjectured polygon P' is in fact P, for if $P' \neq P$, we can continue to look for invisible vertices, secure that we eventually must find them.

In this section, we approach the verification problem by considering static, rather than interactive probing strategies. While these do not lead to full verification strategies, they provide interesting bounds on the size of any conjectured $P' \neq P$ which is consistent with the results of all previous probes. Proofs will appear in the full paper.

Lemma 3 *If two polygons P_1 and P_2, each with at most n vertices, are consistent with the results of any $2n$ parallel x-ray probes, then P_1 and P_2 have exactly the same vertex set.*

Theorem 4 *If two polygons P_1 and P_2, each with at most n vertices, are consistent with the results of any $n(n-1)/2+1$ parallel x-ray probes, then P_1 and P_2 are identical.*

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