On General Properties of Strictly Convex Smooth Distance Functions in $\mathbb{R}^D$

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Abstract

We investigate properties of bisectors based on strictly convex smooth distance functions in $\mathbf{R}^D$ and give methods for constructing bisectors as well as their intersections. An example is sketched to show that in 4-space the intersection of bisectors for 4 points may consist of many disjoint curves. Using Thom's Transversality Lemma we then describe structural properties of intersections of bisectors that are satisfied by "most" sets of point sites; this result provides a formal justification for arguments frequently used in works on Voronoi diagrams, namely, arguments based on "general position" or "non-degeneracy" assumptions.

1 Introduction

It is interesting to observe that in works on Voronoi diagrams based on convex distance functions in higher-dimensional space no reference is given for the basic facts about geometrical structure of bisectors and their intersections (e.g., Are they regular connected surfaces? Do they have the same properties as in the Euclidean metric? Do the bisectors for three points cross each other properly?); of course, these basic facts are important in computing Voronoi diagrams. In this paper we prove results on the geometrical structure of bisectors and their intersections in smooth strictly convex distance functions.

Further, "non-degeneracy" or "general position" assumptions for Voronoi-diagrams are often made in the literature. But (except in the Euclidean metric) the formal justification for such assumptions, to our knowledge, are still lacking. Moreover, as far as we know, even adequate *formulation* of "general position" assumptions in higher-dimensional Euclidean space (again except in the Euclidean metric) are not known. The reason seems to lie in the fact that as soon as one leaves the Euclidean metric the bisectors are *no more* linear spaces. In this paper we formulate the notion of "non-degeneracy" and give a formal justification for it. Our proof is based on an important tool from the theory of singularities of mappings, namely, the Thom's Transversality Lemma.

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2 Basic definitions and results

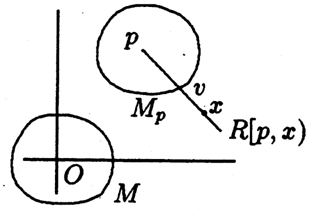

Figure 1: Defining d_M-distance from x to p.

Let $M \subset \mathbf{R}^D$ be the boundary of a compact and strictly convex set containing the origin O in its interior. Using M we define a mapping $d_M : \mathbf{R}^D \times \mathbf{R}^D \longrightarrow \mathbf{R}$ as follows. For all pair of distinct points p and x in $\mathbf{R}^D$, let M_p be the image of M under the translation with the translation vector p; that is, $M_p = M + p$. Let $v = R[p,x) \cap M_p$, where $R[p,x)$ denotes the ray (halfline) from p through x (Fig. 1). We set

$$d_M(x,p) = \frac{|xp|}{|vp|}.$$

If $x = p$, then we set $d_M(x,p) = 0$. The mapping d_M is a *distance function* on $\mathbf{R}^D$. Let p and q be distinct points in $\mathbf{R}^D$. The set $B(p,q) = \{x \in \mathbf{R}^D \mid d_M(x,p) = d_M(x,q)\}$ is called the *bisector* of p and q (with respect to d_M, or just w.r.t. M).

A *supporting hyperplane* of a convex set Q is a hyperplane L containing a point of Q and such that the interior of Q lies entirely in one half-space bounded by L. A convex set $Q \subset \mathbf{R}^D$ is called *smooth* if its boundary hypersurface is smooth.

Let $M \subset \mathbf{R}^D$ be the boundary of a smooth and strictly convex set Q. For each $u \in M$ let L_u be the supporting hyperplane of Q containing u. The outward normal vector of M at u is the unique element $n \in S^{D-1}$ (the unit sphere of $\mathbf{R}^D$) such that $n \perp L_u$ and $n \cdot \overrightarrow{tu} > 0$ for some point t in the interior of Q (Fig. 2). The *normal mapping* $N : M \longrightarrow S^{D-1}$ is defined by $u \longmapsto N(u) := n$; see [4], Theorem 6.2.2. The following result relates M to the sphere S^{D-1}.

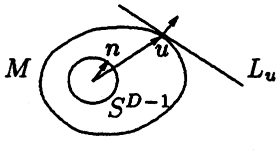

Figure 2: Defining the normal mapping.

Theorem 1 *(Hadamard) The normal mapping of M is a diffeomorphism.* □

Let L be a linear subspace of $\mathbf{R}^D$ whose dimension is $K \leq D-1$. By $\Sigma_L(M)$ we denote the set of all points u of M such that the tangent to M at u is parallel to the linear space L; see Fig. 3.

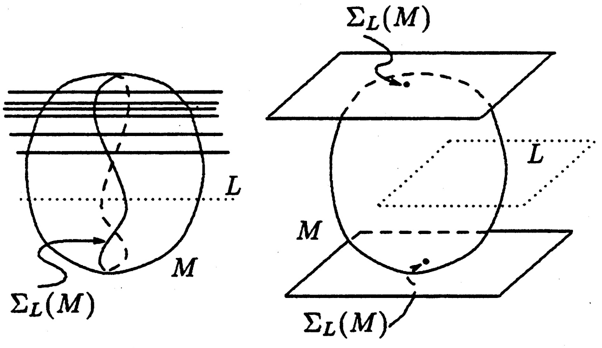

Figure 3: Examples for the set $\Sigma_L(M)$.

Lemma 2 *The set $\Sigma_L(M)$ is homeomorphic to the sphere S^{D-K-1} and a submanifold of M.*

Proof. The set $\Sigma_L(S^{D-1})$ is the intersection of the linear space that is complementary and normal to L and passes through the center of S^{D-1}. Hence $\Sigma_L(S^{D-1})$ is S^{D-K-1}. By Theorem 1 we have $\Sigma_L(M)$ is the pre-image of $\Sigma_L(S^{D-1})$ under the normal mapping N. □

Note that S^0 consists of two points.

3 Bisectors, their tangents, and their intersections

Let δ be a vector in $\mathbf{R}^D$. We define the *dark side* of M *(in direction δ)*, denoted by $\Phi_\delta(M)$, the set of all points $u \in M$ satisfying *1.* the line passing through u and parallel to δ intersects M in a second point $u' \neq u$, and *2.* the vectors $\overrightarrow{u'u}$ and δ have the same direction.

Let pq denote the line connecting p and q. Then clearly the boundary of the dark side of M is the set $\Sigma_{pq}(M)$ defined above, here with $L = pq$.

Lemma 3 *The set of unit outer normals to the dark side of M in direction δ is the dark side of the sphere S^{D-1} in that direction; that is, $N(\Phi_\delta(M)) = \Phi_\delta(S^{D-1})$.* □

3.1 Constructing bisectors and tangents to them

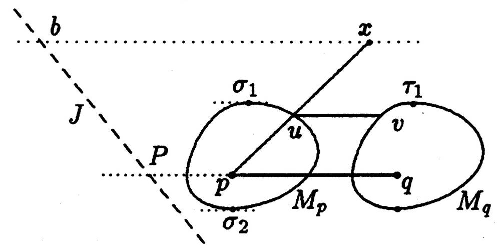

Figure 4: Constructing $B(p,q)$.

The following result shows that the bisector $B(p,q)$ and the dark side of M in the direction $\overrightarrow{pq}$ are diffeomorphic. Let $\delta = \overrightarrow{pq}$.

Lemma 4 *For each $q \in \mathbf{R}^D$ with $q \neq p$ there exists a diffeomorphism $\varphi_p(\cdot, q) : \Phi_\delta(M_p) \longrightarrow B(p,q)$.*

Proof. Let u be any point in $\Phi_\delta(M_p)$. The line through u parallel to δ intersects $\Phi_{-\delta}(M_q)$ in a point v. Let $x \in \mathbf{R}^D$ be given by

$$\overrightarrow{px} = \overrightarrow{pu}/(1 - |uv|/|pq|) \qquad (1)$$

We define the required mapping $\varphi_p(\cdot, q) : \Phi_\delta(M_p) \rightarrow B(p,q)$ by setting $\varphi_p(u,q) = x$. First we have to prove that $x \in B(p,q)$. From (1) we have $\overrightarrow{px} - (|uv|/|pq|) \cdot \overrightarrow{px} = \overrightarrow{pu}$. It follows that $\overrightarrow{ux} = (|uv|/|pq|) \cdot \overrightarrow{px}$. So x, v, q are collinear and $v = R[q,x) \cap M_q$ (recall that $R[q,x)$ denotes the ray from q through x). From $|xp|/|up| = |xq|/|vq|$ follows $d_M(x,p) = d_M(x,q)$.

Next we show that $B(p,q)$ is parameterized by φ_p. For each $x \in B(p,q)$ let $u = R[p,x) \cap M_p$ and $v = R[q,x) \cap M_q$. Then $|xp|/|up| = |xq|/|vq|$, so $u \in \Phi_\delta(M_p)$ and $v \in \Phi_{-\delta}(M_q)$. Hence $\overrightarrow{px} = (|pq|/|uv|) \cdot \overrightarrow{ux}$; i.e., $\overrightarrow{px} = (|pq|/|uv|) \cdot (\overrightarrow{up} - \overrightarrow{px})$. This expression is (1). Finally we prove that φ_p is smooth. In fact, since the scalar- and vector-valued functions $|uv|$ and $\overrightarrow{pu}$ in Formula (1) are smooth, the vector-valued function $\overrightarrow{px}$ is also smooth. Using polar coordinates at p, the set $B(p,q)$ is the graph of φ_p. In other words, the map φ_p is a diffeomorphism. □

We need the following result in the plane.

Theorem 5 *Let $M \subset \mathbf{R}^2$ be the boundary of a strictly convex smooth set containing the origin O in its interior. Let p, q and r be distinct points in $\mathbf{R}^2$. Then*

1. If p, q, r are collinear then $B(p,q) \cap B(q,r) = \emptyset$.

2. *If p, q, r are not collinear then $B(p,q) \cap B(q,r)$ consists of a single point.*

3. *Let σ_1, σ_2 denote the points of $\Sigma_{pq}(M_p)$ and τ_1, τ_2 the points of $\Sigma_{pq}(M_q)$. Then the bisector $B(p,q)$ is fully contained in the interior of the strip whose boundary consists of the rays $R[p,\sigma_1)$, $R[p,\sigma_2)$, $R[q,\tau_1)$ and $R[q,\tau_2)$; see Fig. 4.*

4. *Let J be a line properly crossing the line pq, then $B(p,q)$ is the graph of a real-valued continuous function f defined on J; i.e., $B(p,q) = \{(x, f(x)) \mid x \in J\}$.*

Proof. For the proof of Claim *1* see [3, Theorem 3] and for Claim *2* and *3* see [3, Lemma 1]. We prove Claim *4*. Consider the projection along pq from $\mathbf{R}^2$ onto J; let us denote by π the restriction to $B(p,q)$ of this mapping. The mapping f stated in the Claim will be defined as the "height function" on J (along the direction pq) of the inverse of π. To this end, we must show that π is injective, its inverse is continuous and has whole J as domain.

To show that π is injective we prove that each line parallel to pq intersects $B(p,q)$ in at most one point; we omit the simple proof.

Next we show that π^{-1} is continuous. Consider the composite mapping $\pi \circ \varphi_p : \Phi_\delta(M_p) \longrightarrow J$. Since φ_p is a homeomorphism and π is an injective and continuous mapping, the composite mapping $\pi \circ \varphi_p$ is injective and continuous. Thus, by the Domain Invariance Theorem [6, Cor. 3.22], its inverse is continuous. Clearly, it follows that π^{-1} is continuous.

From above we know more, namely, the mapping $\pi \circ \varphi_p$ is a homeomorphism. We conclude that the range of $\pi \circ \varphi_p$ is homeomorphic to $\Phi_\delta(M_p)$. Thus, the range of π is an open interval; we show that it has no bounded ends. Let b be the projection along pq of $x \in B(p,q)$ on J (Fig. 4). From Formula (1) it follows that $|px| \to \infty$ as u approaches to any point of $\Sigma_{pq}(M_p)$; moreover from Claim 3 we know that the point x must lie within a strip with bounded width. Therefore we have $\lim |Pb| = \infty$ as u approaches σ_1 or σ_2, where $P := pq \cap J$. □

Now we return to the higher-dimensional case. The next two lemmas generalize well-known facts in the Euclidean metric.

Lemma 6 *[3] For each point $x \in B(p,q)$ the tangent hyperplane to $B(p,q)$ at x is the linear space spanned by x and the intersection of the tangent hyperplanes of M_p at u and M_q at v, where $u := R[p,x) \cap M_p$ and $v := R[q,x) \cap M_q$.* □

Let M^R be the reflection of the sphere M about its center. For each point $x \in B(p,q)$ let M' be the sphere homothetic to M^R with center x (i.e., $M' = \lambda M^R + x$ for some $\lambda > 0$) and passes through p and q. The existence and uniqueness of M' follows immediately from the definition of $B(p,q)$.

Lemma 7 *The tangent hyperplane to $B(p,q)$ at x is the linear space spanned by x and the intersection of the tangent hyperplanes of M' at p and q.*

Proof. We consider in addition the sphere M_p^R, the reflection of M_p about its center p; see Fig. 5. The ray from x to p intersects M_p at u and M_p^R at p'. We claim that $T_u M_p$ and $T_p M'$ are parallel. In fact, we first see that $T_u M_p$ and $T_{p'} M_p^R$ are parallel since they are reflections of each other about p. Next observe that $T_p M'$ and $T_{p'} M_p^R$ are parallel because the homothety with center c (Fig. 5) transforming M' to M_p^R sends p to p', and hence takes M' to M_p^R. Analogously we can also prove that $T_v M_q$ and $T_q M'$ are parallel.

Since $x \in B(p,q)$ we have $|xp|/|up| = |xq|/|vq|$. Consider the homothety with center x that sends u to p, and v to q. It takes $T_u M_p$ to $T_p M'$ since they pass through u and p, respectively, and we know from above that these hyperplanes are parallel. By analogous arguments this homothety takes $T_v M_q$ to $T_q M'$. Clearly it transforms the intersection of $T_u M_p$ and $T_v M_q$ to the intersection of $T_p M'$ and $T_q M'$. This fact together with Lemma 6 imply the Claim. □

Now we are ready to prove the main result of this subsection. In the Euclidean metric the bisector is a hyperplane. We show that an analogy still holds for smooth strictly convex-distance functions.

Theorem 8 *Let $\mathcal{H}$ be a hyperplane properly crossing the line pq. Then the bisector $B(p,q)$ is the graph of a smooth real-valued function defined on $\mathcal{H}$.*

Proof. Consider the projection along pq that maps $\mathbf{R}^D$ onto $\mathcal{H}$; let π be the restriction to $B(p,q)$ of this mapping. We will prove that π is injective, its inverse is continuous and has $\mathcal{H}$ as its domain.

Observe that the mapping π leaves invariant any linear space $\mathcal{L}$ that is parallel to the projection direction pq. Hence, to reduce the problem to 2-dimensional ones, we scan $\mathbf{R}^D$ by rotating a plane $\mathcal{L}$ around pq that can be taken as the linear space spanned by a point v in $\Sigma_{pq}(M_p)$ and the line pq. In fact, letting v varies in $\Sigma_{pq}(M_p)$ then the set swept out by $\mathcal{L}$ covers the whole space $\mathbf{R}^D$.

The restriction of π to $\mathcal{L} \cap B(p,q)$, denoted by π^*, maps $\mathcal{L} \cap B(p,q)$ into the line $\mathcal{L} \cap \mathcal{H}$. To show that π is surjective, it suffices to show that π^* is bijective. Observing that $\mathcal{L} \cap B(p,q)$ is the bisector of p and q with respect to the smooth strictly convex *curve* $\mathcal{L} \cap M_p - p$, the bijectivity of π^* follows from Claim *4* of Lemma 5. We conclude that the continuous mapping π is bijective.

Now consider the mapping $\pi \circ \varphi : \Phi_\delta(M_p) \to \mathcal{H}$, which is the composition of $\varphi : \Phi_\delta(M_p) \to B(p,q)$ (given in Lemma 4) and π. We know that φ is a homeomorphism and that, from above, π is continuous and bijective. By the Domain Invariance Theorem [6, Corollary 3.22] it follows that the inverse mapping of $\pi \circ \varphi$ is continuous. Clearly the inverse of π is continuous. Hence the bisector $B(p,q)$ is the graph of a continuous real-valued function defined on $\mathcal{H}$. Using Lemma 6 we see that at each point $x \in B(p,q)$ the tangent to $B(p,q)$ must cross the line pq properly. Hence at each $x \in B(p,q)$ the differential of π is regular. The smoothness of π^{-1} follows. □

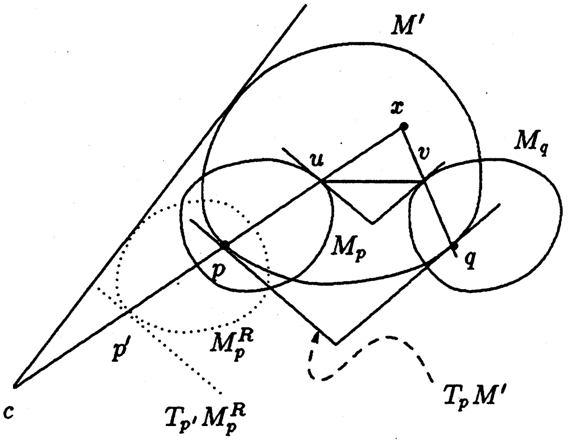

Figure 5: Tangent to bisector.

3.2 Intersection of bisectors

We say that two smooth submanifolds X_1 and X_2 of a smooth manifold Y *intersect transversely (in Y)* if at every $x \in X_1 \cap X_2$ the tangent spaces T_xX_1 and T_xX_2 intersect transversely in T_xY; i.e., $T_xX_1 + T_xX_2 = T_xY$; we write then $X_1 \pitchfork X_2$.

Theorem 9 *The bisectors $B(p,q)$ and $B(p,r)$ for three distinct points p,q,r intersect transversely whenever they intersect at all.*

Proof. For each $x \in B(p,q,r)$ let M' be the sphere homothetic to M^R with center x that passes through p,q and r. Now observe that since M' is strictly convex the tangents to M' at the distinct points p,q and r do not intersect in a linear space of dimension $D-2$. It follows from Lemma 7 that the tangent hyperplanes to $B(p,q)$ and $B(p,r)$ must intersect properly. □

Since the transverse intersection of two manifolds is a manifold, Theorem 9 implies that the intersection of two bisectors for three points is a manifold. At this point we don't know more about the exact structure of $B(p,q) \cap B(p,r)$. In the Euclidean metric the bisectors for three non-collinear points intersect in a linear space of dimension $D-2$; the following results confirms that in smooth strictly convex distance functions the analogy to the Euclidean metric still holds. Before stating the result mentioned we need some facts. Let P be a set of K points spanning a linear space $L(P)$ of dimension $K-1$; further assume that $K \le D$. Other cases can be solved using this basic case. Then by Lemma 2 the set $\Sigma_{L(P)}(M_p)$ is of dimension $D-K$. For simplicity we will write Σ_P instead of $\Sigma_{L(P)}$. For each $u \in \Sigma_P(M_p)$ the segment $[p,u)$ lies in the interior of M_p; we denote by $p\Sigma_P(M_p)$ the union of all such segments. The *cone* $p\Sigma_P(M_p)$ is homeomorphic to the unit ball of $\mathbf{R}^{D-K+1}$ and a submanifold of $\mathbf{R}^D$.

Theorem 10 *Let $D \ge 3$. The intersection $B(p,q,r)$ of the bisectors $B(p,q)$ and $B(q,r)$ is a $D-2$ dimensional manifold that is homeomorphic to a $D-2$ dimensional linear space.*

Proof. Let $P = \{p,q,r\}$. The manifold $\Sigma_P(M_p)$ has the dimension $D-3$ (Lemma 2). Hence the cone $p\Sigma_P(M_p)$ has the dimension $D-2$. Consider any point t of the cone $p\Sigma_P(M_p)$; see Fig. 6. Through t there exists exactly one plane $\pi(t)$ passing through t and parallel to the linear space $L(P)$. Each $\pi(t)$ intersects M_p in a smooth strictly convex curve $N_p(t)$. Let $z(t)$ be the intersection of the bisectors $B(t,q(t))$ and $B(q(t),r(t))$ with respect to the distance function defined by $N_p(t)$ with center t (Theorem 5, Claim 2), where the points $q(t)$ and $r(t)$ are defined analogously as t. Determine $u(t)$ and then $x(t)$ as shown in Fig. 6, then $x(t) \in B(p,q,r)$. Conversely, if $x \in B(p,q,r)$ then x can be parameterized in the way just described by exactly one point $t \in p\Sigma_P(M_p)$ and one point $u(t)$. We omit the details. □

We now sketch how to extend the approach above to a general scheme for constructing the intersection of bisectors. Let $B(P) = B(p_1,p_2) \cap B(p_2,p_3) \cap \cdots \cap B(p_{K-1},p_K)$. The linear space $L(P)$ has the dimension $K-1$, hence the manifold $\Sigma_P(M_p)$ has the dimension $D-K$. So the cone $p\Sigma_P(M_p)$ has the dimension $D-K+1$. There is exactly one linear space of dimension $K-1$ passing through t, and parallel to $L(P)$; it intersects M_p in a $K-2$ dimensional smooth manifold $N_p(t)$. Consider $Z(t)$, the set of all $z(t)$ determined analogously as in the proof of Theorem 10. Note that $Z(t) \ne \emptyset$ (see [5]), and in general $|Z(t)|$ may be > 1 (see [3]). In this way we can parameterize the set $B(P)$. We omit the details.

In the special case when all the slices $N_p(t)$ are homothetic and $U(t)$ consists of finitely many points, then $B(P)$ consists of $|U(t)|$ manifolds each of which has the same dimension as the cone $p\Sigma_P(M_p)$.

Lemma 11 *In 4-space there exist four points p,q,r,s and a smooth strictly convex distance function such that the intersection $B(p,q) \cap B(q,r) \cap B(r,s)$ consists of three curves each of which is homeomorphic to a line.* □

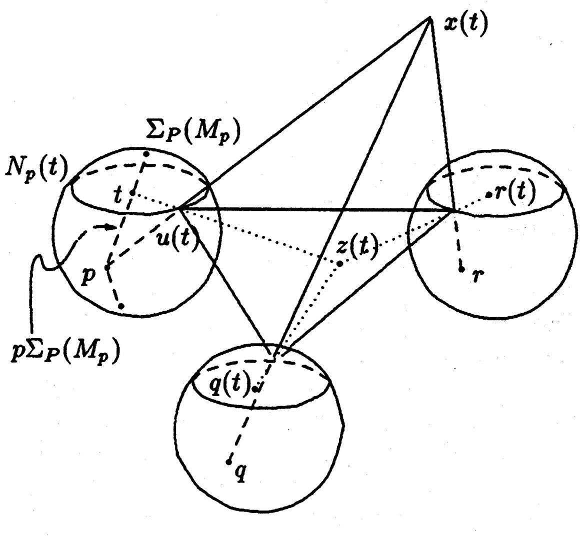

Figure 6: Constructing $B(p_1, \ldots, p_K)$.

4 The notion of "in general position"

To avoid cumbersome notations we fix the dimension $D = 3$. The results in this section hold also in the general case. To illustrate our key idea for the proof of the results in this section, let us consider two smooth curves X and Y in the plane. Even when the curves do not intersect transversely, intuitively we feel that by translating X an arbitrarily small amount, we can "always" force them to do so. In fact, this intuition is verified by first proving that the mapping $f : X \times \mathbf{R}^2 \to \mathbf{R}^2$ defined by $f(x, v) = x + v$ is a *submersion* and then applying Thom's Transversality Lemma to f.

Let X and Y be smooth manifolds. Let $f : X \to Y$ be a smooth mapping. We call f a *submersion* if at each $x \in X$ the linear map $(df)_x : T_xX \to T_{f(x)}Y$ is surjective.

Let $f : X \to Y$ be a smooth mapping. Let W be a smooth manifold of Y. Then f is *transverse* to W if for all $x \in X$ either *1.* $f(x) \notin W$, or *2.* $f(x) \in W$ and $T_{f(x)}W + (df)_x(T_xX) = T_{f(x)}Y$.

Theorem 12 *(Thom's Transversality Lemma [2, p. 79]) Let X, B and Y be smooth manifolds. Let $W_1, \ldots, W_K$ be smooth submanifolds of Y. Let $f : X \times B \to Y$ be a smooth mapping. Assume that f is transverse to all $W_1, \ldots, W_K$. Then for almost all $b \in B$ and for $i = 1, \ldots, K$ the submanifolds $f(X, b)$ and W_i intersect transversely, where $f(X, b)$ is the image of X under $f(\cdot, b)$.* □

We will also need the following result:

Lemma 13 *Let $f : X \to Y$ be a submersion. Then f is transverse to every submanifold of Y.* □

Our plan is now clear: We first investigate in Subsection 4.1 how the bisector $B(p, q)$ changes while keeping a point site fixed and letting the other one varying; in fact, we will show an analogy to the case of translating curves discussed before. Finally, using this result and Thom's Transversality Lemma we show in Subsection 4.2 that an analogous statement holds for system of bisectors, when the point sites are considered as parameters.

4.1 Bisector as function of point site

Let q_0 be a fixed point with $q_0 \neq p$. Let n and n_0 be the intersection of S_p^2 and the halfline from p through q and q_0 respectively. Let α be the angle with $\overrightarrow{pn_0}$ as the initial side and $\overrightarrow{pn}$ as the terminal side. Let $Q = \{n_0, n_1\}$. Define the rotation axis $a = \overrightarrow{pn_0} \times \overrightarrow{pn}$, and let $R(a; \alpha) : S_p^2 \longrightarrow S_p^2$ be the rotation on the sphere around a with angle α. We use the mappings φ_p and N (see Section 2 and Subsection 3.1, here we write φ without subscript p) to define a mapping τ: $\tau(\cdot, n) = \varphi(\cdot, n) \circ N^{-1} \circ R(a; \alpha) \circ N \circ \varphi(\cdot, q_0)^{-1}$. Finally define $h : B(p, q_0) \times (\mathbf{R}^3 \setminus pq_0) \longrightarrow \mathbf{R}^3$ by $h(x, q) = p + |pq| \cdot (\tau(x, p + \overrightarrow{pq}/|pq|) - p)$. We now state without proof a key result.

Theorem 14 *The mapping $h : B(p, q_0) \times (\mathbf{R}^3 \setminus pq_0) \longrightarrow \mathbf{R}^3$ is a submersion. Moreover, the image of $B(p, q_0)$ under $h(\cdot, q)$ is exactly $B(p, q)$; that is, $h(B(p, q_0), q) = B(p, q)$.* □

4.2 Configurations of point sites in general position are dense

Let S be a set of N distinct points $p_1, \ldots, p_N \in \mathbf{R}^3$. The N-tuple $(p_1, \ldots, p_N)$ is called an *N-configuration.* We say that the point sites $p_1, \ldots, p_N$ are *non-degenerate* or *in general position* if for any subsets T of S, any $q \in S$ but $\notin T$, and any $t \in T$ the sets $B(T)$ and $B(t, q)$ intersect transversely. An N-configuration is *non-degenerate* or *in general position* if its components are. All N-configurations with the same set of components are considered equivalent.

Let X and Y be submanifolds of $\mathbf{R}^D$. If X and Y intersect transversely then $X \cap Y$ is a submanifold whose dimension is $\dim X + \dim Y - D$, whenever $X \cap Y \neq \emptyset$. Moreover, observe that if X and Y intersect transversely and if $\dim X + \dim Y < D$ then they must have empty intersection. Using these facts we can prove the following lemma.

Lemma 15 *The distinct points $p_1, \ldots, p_N$ are non-degenerate if and only if*

1. *$B(p_i, p_j) \pitchfork B(p_i, p_k)$, for all distinct i, j, k*
2. *$B(p_i, p_j, p_k) \pitchfork B(p_i, p_l)$, for all distinct i, j, k, l*

3. $B(p_i,p_j,p_k,p_l) \pitchfork B(p_i,p_m)$, *for all distinct* i,j,k,l,m.

Relation 3 can be replaced by

3'. $B(p_i,p_j,p_k,p_l) \cap B(p_i,p_m) = \emptyset$ *for all distinct* i,j,k,l,m.

(Note that on both sides of each relation $\pitchfork$ *above there is always a point site in common.)* □

Note that Relation *3* or equivalently *3'* says that no five points lie on a sphere homothetic to M^R. The notion of "in general position" or "non-degenerate" is justified by the following theorem.

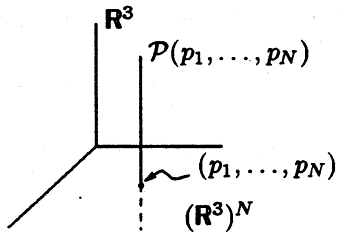

Figure 7: Defining the set $G^{N+1} \subset (\mathbf{R}^3)^{N+1}$.

Theorem 16 *The set* $G^N \subset (\mathbf{R}^3)^N$ *of non-degenerate N-configurations is dense in* $(\mathbf{R}^3)^N$. *Moreover, the complementary set of* G^N *in* $(\mathbf{R}^3)^N$ *has measure zero, if it is measurable.*

Proof. The proof proceeds by induction on N. For $N = 3$ the theorem is true by Theorem 9. Assuming the theorem for N, we prove it for $N+1$.

For each $(p_1,\ldots,p_N) \in G^N \subset (\mathbf{R}^3)^N$ we will construct a residual set $\mathcal{P}(p_1,\ldots,p_N)$ in $\mathbf{R}^3$ (for the notion *residual* see [2]) consisting of points q so that the configuration $(p_1,\ldots,p_N,q)$ is non-degenerate; we then obtain the required set G^{N+1} as the union of the sets $\mathcal{P}(p_1,\ldots,p_N)$ when the N-tuple $(p_1,\ldots,p_N)$ varies in G^N; see Fig. 7.

Consider any $(p_1,\ldots,p_N) \in G^N \subset (\mathbf{R}^3)^N$. We first construct a set $\mathcal{P}^{(1)}$ with the property that Relations *1*, *2* and *3* of Lemma 15 hold under the condition that the common point site is p_1. By the induction hypothesis, for all distinct $j,k,l \le N$ but $\ne 1$, the sets $B(p_1,p_j,p_k), B(p_1,p_j,p_k,p_l) \subset \mathbf{R}^3$ are 1-, and 0-manifolds. By Theorem 14 and Lemma 13 we can apply Thom's Transversality Lemma (Theorem 12) with $X = B(p_1,q_0), B = \mathbf{R}^3 \setminus p_1q_0, Y = \mathbf{R}^3, f = h$ and the W_i's are the manifolds $B(p_1,p_j), B(p_1,p_j,p_k), B(p_1,p_j,p_k,p_l) \subset \mathbf{R}^3$ to obtain a residual set $\mathcal{P}^{(1)} \subset \mathbf{R}^3$ such that for all q in that set we have

$$\begin{array}{lll} B(p_1,p_j) & \pitchfork & h(B(p_1,q_0),q) \\ B(p_1,p_j,p_k) & \pitchfork & \cdots \\ B(p_1,p_j,p_k,p_l) & \pitchfork & \cdots \end{array}$$

By Theorem 14 we have $h(B(p_1,q_0),q) = B(p_1,q)$. Hence

$$\begin{array}{lll} B(p_1,p_j) & \pitchfork & B(p_1,q) \\ B(p_1,p_j,p_k) & \pitchfork & \cdots \\ B(p_1,p_j,p_k,p_l) & \pitchfork & \cdots \end{array}$$

We consider the next site p_2 and repeat the procedure above for p_2, but now with a smaller set of sites $S \setminus \{p_1\}$. In this way we obtain a residual set $\mathcal{P}^{(2)}$ in $\mathbf{R}^3$ with the property that, for all $q \in \mathcal{P}^{(2)}$, Relations *1*, *2* and *3* are satisfied for p_2 as the common site for both sides of each relation.

Proceeding analogously to above we reach the site p_{N-2}, and obtain a set $\mathcal{P}^{(N-2)}$. Finally we define the set $\mathcal{P}(p_1,\ldots,p_N)$ mentioned at the beginning as $\mathcal{P}(p_1,\ldots,p_N) = \bigcap_{i=1,\ldots,N-2} \mathcal{P}^{(i)}$. The set $\mathcal{P}(p_1,\ldots,p_N)$ is residual. By construction, for all $q \in \mathcal{P}(p_1,\ldots,p_N)$, the point sites $p_1,\ldots,p_N$ and q are non-degenerate, since Relations *1*, *2* and *3* in Lemma 15 hold for them. We then define the required set G^{N+1} by taking the union of all $\mathcal{P}(p_1,\ldots,p_N)$ while letting $(p_1,\ldots,p_N)$ varies over G^N; i.e., $G^{N+1} = \bigcup_{(p_1,\ldots,p_N)\in G^N} \mathcal{P}(p_1,\ldots,p_N)$. Clearly the set G^{N+1} is dense in $(\mathbf{R}^3)^{N+1}$.

Claim *2* follows from Fubini-Tonelli Theorem. □

5 Concluding Remarks

One may ask if the set $G^N \subset \mathbf{R}^N$ of non-degenerate N-configurations is *open* and Voronoi diagrams of N-configurations belonging to the same connected component of G^N are combinatorially equivalent. Positive results for this question seem to be important for ensuring the stability of future numerical computations of Voronoi diagrams based on smooth strictly convex distance functions. Another interesting problem is to bound the number of connected components of G^N.

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