Near-Quadratic Bounds for the L_1 Voronoi Diagram of Moving Points

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Abstract

Given a set of n moving points in the plane, how many topological changes occur in the Voronoi diagram of the points? If each point has constant velocity then there is an upper bound of $O(n^3)$ [Guibas, Mitchell, and Roos] and an easy lower bound of $\Omega(n^2)$. It is widely believed that the true upper bound should be close to $O(n^2)$. We show this belief to be true for the case of Voronoi diagrams based on the L_1 (or L_∞) metric; the number of changes is shown to be $O(n^2\alpha(n))$ where $\alpha(n)$ grows so slowly it is effectively a small constant for all reasonable values of n.

1 Introduction

Suppose we have a set of points in the plane, each point moving with its own, constant velocity. Consider the Voronoi diagram of the points; this diagram moves continuously as the points move, but distinct topological changes occur when 4 points become cocircular. The question we address is: How many such topological changes can occur? For constant-velocity points in the plane, it is widely suspected that the number of topological changes in the Voronoi diagram is near $O(n^2)$. This suspicion is presently supported by a lack of counterexamples rather than a proof; the best lower bound is currently $\Omega(n^2)$ which can be achieved by two lines of points passing each other in opposite directions as if on a roadway. The best upper bound is currently $O(n^3)$ [GMR91], proved using the technique of linearization. In this paper we show that for Voronoi digrams based on the L_1 (or L_∞) metric, the bound on the number of topological changes as the source points move is near $O(n^2)$, namely $O(n^2\alpha(n))$ where α is the very slowly growing inverse Ackermann's function.

The technique used here is related to a technique used by the author and K. Kedem in [CK89, CK93] where a type of Voronoi diagram is used for placing a convex object among

*This work was supported by the Advanced Research Projects Agency of the Department of Defense under ONR Contract N00014-92-J-1989, and by ONR Contract N00014-92-J-1839, NSF Contract IRI-9006137, and AFOSR Contract AFOSR-91-0328.

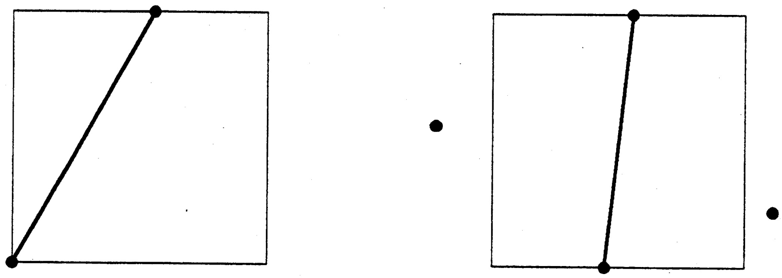

Figure 1: A corner edge and a noncorner edge.

polygonal obstacles. The number of topological changes that occur in a convex-distance-function-based Voronoi diagram as the distance defining shape (a convex polygon with k sides) is rotated was shown to be $O(k^4n^2\alpha(n))$.

In some ways it is more natural to work with the *Delaunay triangulation*, the dual of the Voronoi diagram. Topological changes in the Voronoi diagram correspond to edge flips in the Delaunay triangulation. Adjacencies in the Delaunay triangulation remain fixed except at distinct events when the four points of two adjacent Delaunay triangles become cocircular.

Given a set of moving points, the number of changes that occur in the Delaunay triangulation appears to be closely related to the problem of counting the number of changes that can occur in the Minimum Spanning Tree (MST), since it is well known that the MST is a subgraph of the Delaunay triangulation. However, there are changes in the MST that do not correspond to changes in the Delaunay triangulation – edge lengths in the Delaunay triangulation can change enough to alter the MST without causing a change in the Delaunay triangulation. Katoh, Tokuyama, and Iwano [KTI92] have shown that for the L_1 (or L_∞) distance and for n points, each moving at constant velocity, the MST undergoes $O(n^{\frac{5}{2}}\alpha(n))$ changes.

2 Counting Corner Edge Changes

We assume the reader is familiar with Delaunay triangulations. In particular, we make use of the fact that for each edge of the Delaunay triangulation there is an empty circle that goes through the endpoints of the edge. For the L_∞ metric, the *circle* that we use is actually a square. For the L_1 metric, the *circle* is a square tipped at 45 degrees. For our problem the L_1 and the L_∞ metric are equivalent since only the shape matters. For the remainder of this paper, we use the L_∞ metric since it is easier to draw squares that are not tipped.

We distinguish two types of Delaunay edges: *corner edges* and *noncorner edges* (see Figure 1). A Delaunay edge is called a corner edge if there exists an empty square through the endpoints of the edge with a corner of the square on one of the endpoints. The remaining Delaunay edges are called noncorner edges. Note that since an empty square placed on an edge can slide while remaining in contact with the edge's endpoints, a Delaunay edge is a noncorner edge only if the sliding is blocked by other source points (see Figure 1) – otherwise the square could slide far enough to show that the edge is really a corner edge. Each noncorner

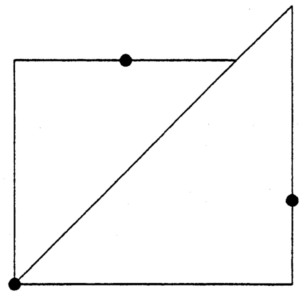

Figure 2: The closer hit corresponds to a corner edge.

edge has a pair of these *blocking points.*

Our goal is to determine the number of changes in the Delaunay triangulation as the source points move. Any change in the Delaunay triangulation is a change in either a corner edge or a noncorner edge. If we can determine a bound on these edge changes then we also have a bound on the total number of Delaunay triangulation changes. We start by bounding the number of changes for corner edges.

Theorem 1 For n points, each moving with constant velocity, there are $O(n^2\alpha(n))$ changes in the set of corner edges.

Proof: We show that for each source point p, there are $O(n\alpha(n))$ changes in its corner edges. We then sum over all points to get the bound in the theorem. Note that there are four types of corner edges corresponding to the four corners of the square. Without loss of generality, we consider just the corner edges corresponding to the lower left corner of the square.

At a fixed time t, we can find the corner edge for source point p by placing a very small square with its lower left corner on p. We then grow the square until it hits some other source point. This first point that we hit (assuming we hit something at all) determines the current corner edge for p. We recast this view of corner edges to use triangles instead of a square. We split the square into two triangles (see Figure 2). Think of placing both triangles on our source point p, then expanding each of them independently until they each hit a source point. The closer of the (at most) two hits determines the corner edge for p.

Now consider just one of the two triangles, say the top one. For each source point q ($\neq p$), we create a copy of the triangle and place the lower corner of the triangle on p and expand the triangle until it touches q. We associate a value with each source point q; the value for q is the size of the resulting triangle (measured as the length of the triangle's hypotenuse, for instance). Note that for many source points the expanding triangle does not ever hit them so the associated value is infinity.

As time progresses, the value associated with q changes, so we have a function on t for each q. Note that, for an individual point q, the corresponding function looks like a line segment. The endpoints occur because q can enter and leave the zone where the expanding triangle can touch it. Within this valid zone, the function is linear since q travels with constant velocity with respect to p. Thus, if we plot triangle size for each point q over time,

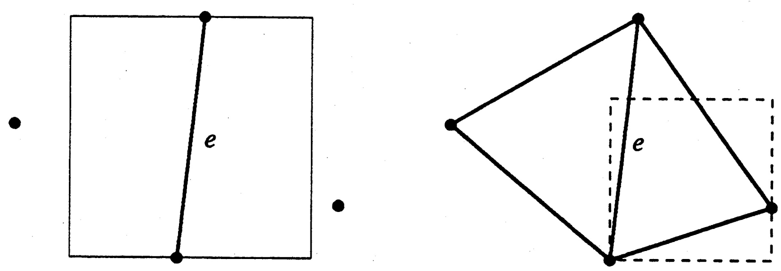

Figure 3: For each noncorner edge e, the endpoints of e and the two blocking points form a cell of four corner edges.

the result looks like a set of at most n line segments. We get a similar set of line segments for the other, lower triangle.

Recall that the corner edge for p corresponds to the first hit we get as we expand the two triangles. In other words, the corner edge for p corresponds to the lower envelope of the combined set of segments from both of the triangles. A change in the corner edge corresponds to a break-point in the lower envelope. The lower envelope for a set of $O(n)$ line segments is of complexity $O(n\alpha(n))$ [ASS89], so there are at most that many changes in the corner edges for point p. The bound in the theorem follows by summing these changes over all four corners of the square and over all points p. □

3 Counting Noncorner Edge Changes

Recall that each change in the Delaunay triangulation corresponds to a change in either a corner edge or a noncorner edge. Now that we have a bound on changes for corner edges, changes for noncorner edges are relatively easy to count.

We claim that each noncorner edge of the Delaunay triangulation exists in a cell consisting of four corner edges. This follows from the earlier observation that, for a noncorner edge, an empty square placed on the edge is blocked from sliding too far by other source points. Once we slide the square against one of these blocking points, it is easy to see that the square can be shrunk to confirm the existence of two corner edges for each blocking point (see Figure 3).

Within the lifetime of a single cell (a quadrilateral) there can be just $O(1)$ swaps of its two diagonals. This is because, with constant velocity points, a set of four points becomes cocircular at most a constant number of times. In total there are $O(n^2\alpha(n))$ cells: $O(n)$ of them existing at time zero and, by Theorem 1, $O(n^2\alpha(n))$ new ones that occur over time as corner edges change. Thus, we have the following theorem:

Theorem 2 For a set of n points, each moving with constant velocity, there are $O(n^2\alpha(n))$ changes in the L_∞ (or L_1) Delaunay triangulation of the point set. The same bound holds for the number of topological changes in the corresponding Voronoi diagram.

4 Discussion

The technique outlined above can be generalized in several ways:

- The technique applies to more than just L_1 and L_∞ distances; it applies to any *convex distance function* where the distance-defining convex shape is a polygon. (For additional information on convex distance functions and their relation to Voronoi diagrams and Delaunay triangulations see, for instance, [CD85] or the survey [Aur91].) The above technique can be used to show that, for n points, each moving with constant velocity, and for a Delaunay triangulation based on a convex distance function determined by a k-sided convex polygon, there are $O(k^4n^2\alpha(n))$ changes in the Delaunay triangulation. I suspect there are too many factors of k in this bound.

- Similar, near-quadratic bounds hold for more complex motions of the source points provided that motion of the points is restricted so that no four points become cocircular more than a constant number of times. Note that *cocircular* here refers to a nonstandard circle.

- Near-quadratic bounds also hold when we use sources more complicated than points. For instance, the sources might be moving rectangles or moving polygons. Techniques are relatively straightforward for polygonal sources that do not rotate and that move in straight lines. Motions that are more complex would have to be restricted to avoid groups of repeatedly cocircular points. See [CK89, CK93] for the definition of the appropriate *edge Delaunay triangulation* and for a partial indication of how more-complex sources could be handled.

- The results presented here can be used to derive a bound on the complexity of moving a (nonrotating) convex polygon among moving obstacles. The use of a type of Voronoi diagram for motion planning is discussed in [CK93] which includes references to earlier, related work.

The work presented here was originally started as part of an attempt to close the complexity gap for changes in the standard Delaunay triangulation of n constant-velocity points. As mentioned in the Introduction, the current lower bound is $\Omega(n^2)$ while the current upper bound is $O(n^3)$. It may be that the techniques presented here will at some point be useful in closing this gap; unfortunately though, there are major differences in the behavior of the standard Delaunay triangulation versus the L_1 Delaunay triangulation. As an example, for an individual source point p among n moving points, our results for L_1 show that there are $O(n\alpha(n))$ changes that can take place in edges that touch p, while for the standard Delaunay triangulation there can be $\Omega(n^2)$ changes.

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