Tighter Bounds on Voronoi Diagrams of Moving Points

(Extended Abstract)

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Abstract

Given a set S of n points in d-dimensional Euclidean space $\mathbb{E}^d$, $d \geq 2$. Thereby $k \leq n$ of them are allowed to move continuously along given trajectories while the remaining $n-k$ sites stay fixed at their position. At each instant, the points of S define a Voronoi diagram which changes continuously except of certain critical instants, so-called topological events. The classification of these events as well as their number has been of very recent interest (cf. [1, 2, 4, 6, 8, 10, 11, 14]). However, up to now there exists a gap of a factor of approximately $\Theta((n-k)^{\lfloor d/2 \rfloor})$ between the best upper bound on the number of topological events and the known worst-case examples. In this paper we present a new upper bound which approaches this lower worst-case bound up to a factor of $O(\min\{k^d, (n-k)^{\lfloor d/2 \rfloor}\})$. This improves the earlier bound in the case of $k \in O(\sqrt{n})$. For the very first time, an approach provides matching upper and lower worst-case bounds (at least in the case of k being some constant).

1 Introduction

The maintenance of geometric data structures over time is the subject of a rising discipline called dynamic computational geometry.[1] The present work investigates the number of topological events which appear in d-dimensional Voronoi diagrams under continuous motions of the underlying sites.

For that, consider a set of n points in d-dimensional Euclidean space, $d \geq 2$. Thereby $k \leq n$ points move along given trajectories and the remaining $n-k$ sites are fixed at their current position. At each instant, the points define a Voronoi diagram which changes continuously except of certain critical instants, so-called *topological events*. The classification of these events has been of recent interest and they are well-understood in the plane [4, 6, 8, 11], in higher-dimensional spaces [1, 2] as well as in planar higher-order Voronoi diagrams [14]. Actually, in all these cases there exist algorithms which maintain the Voronoi diagram in $O(\log n)$ time per event which is not only independent of the dimension and the order of the Voronoi diagram but also worst-case optimal [13].

Up to now, the best upper bound on the number of topological events which appear in classical nearest-neighbor Voronoi diagrams in d-dimensional Euclidean space is $O(\alpha_s(k,n))$ where $\alpha_s(k,n) := k\,n^{d-1}\,\lambda_s(n) + (n-k)^d\,\lambda_s(k)$ (due to [2]). Thereby, $\lambda_s(n)$ denotes the maximum length of a (n,s)-Davenport-Schinzel sequence, and s is a constant depending on the complexity of the underlying trajectories of the moving sites. In opposite to that, the best known worst-case examples generate "only" $O(k\,(n-k)^{\lceil d/2 \rceil})$ topological events. Thus, this leaves a gap of a factor of approximately $\Theta((n-k)^{\lfloor d/2 \rfloor})$.

Now, the main task of this paper is to bring down the upper known worst-case bound to $O(\alpha'_s(k,n))$ topological events where $\alpha'_s(k,n) := \min\{k^{d+1}\,(n-k)^{\lceil d/2 \rceil}, \alpha_s(k,n)\}$. Doing this, the new upper bound now approaches the lower worst-case bound up to a factor of $O(\min\{k^d, (n-k)^{\lfloor d/2 \rfloor}\})$. This improves the previously known bound in the case of $k \in O(\sqrt{n})$.

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[1] In our context the term "kinematic" would be more appropriate. Nevertheless, we use the term "dynamic" for traditional reasons.

The paper is organized as follows. Section 2 introduces the necessary definitions and summarizes the results which are known until now. Afterwards, Section 3 presents the idea which consists in a refined counting argument and the proof of the new upper bound. Finally, Section 4 outlines further research on that topic.

2 Preliminaries

This section briefly summarizes elementary definitions and properties of planar and higher-dimensional Euclidean Voronoi diagrams. At the beginning, we are given a finite set $S := \{P_1, \ldots, P_n\}$ of $n \geq d+2$ points in d-dimensional Euclidean space $\mathbb{E}^d$, $d \geq 2$. (During our investigations, the dimension d is considered to be constant.) The perpendicular *bisector of P_i and P_j* is defined to be the hyperplane $B_{ij} := \{x \in \mathbb{R}^d \,|\, d(x, P_i) = d(x, P_j)\}$, i.e. the set of points with equal distance to P_i and P_j. The (convex) *Voronoi polyhedron of P_i* is given by $v(P_i) := \{x \in \mathbb{R}^d \,|\, \forall_{j \neq i}\ d(x, P_i) \leq d(x, P_j)\}$, i.e. the subset of $\mathbb{E}^d$ which is "dominated" by P_i. As usual, the vertices of the Voronoi polyhedrons are called *Voronoi points* and the bisector portions on the boundary are called *Voronoi l-faces* according to their affine dimension l, for $0 \leq l \leq d$. (We refer to the classification of Voronoi l-faces given in [3].) Finally the *Voronoi diagram of S* can be defined to be the collection of Voronoi polyhedra, i.e. $VD(S) := \{v(P_i) \,|\, P_i \in S\}$. We call the embedding of the Voronoi diagram into the d-dimensional Euclidean space the *geometrical structure* of the underlying Voronoi diagram. The dual graph of the Voronoi diagram is the so-called the *Delaunay graph $DT(S)$*. If S is in general position – i.e. no $d+2$ points of S lie on a common hypersphere and no $d+1$ points of S lie on a common hyperplane – every Voronoi $(d-i)$-face in $VD(S)$ corresponds to an i-face in $DT(S)$, for $i = 0, \ldots, d$.

Next, we introduce a *one-point-compactification* to simplify the description. We augment set S by adding the "point at infinity", yielding a new set of sites $S' := S \cup \{\infty\}$. The extended Delaunay graph is then given by $DT(S') = DT(S) \cup \{(P_i, \infty) \,|\, P_i \in S \cap \partial CH(S)\}$. So, in addition to the Delaunay graph $DT(S)$, every point on the boundary of the convex hull $\partial CH(S)$ is connected to ∞. We call the underlying graph of the extended Delaunay graph $DT(S')$ the *topological structure* of the Voronoi diagram. In contrast with $DT(S)$, $DT(S')$ has the nice property that there are exactly $d+1$ $(d+1)$-tuples adjacent to each $(d+1)$-tuple in $DT(S')$.

Next, we adopt two functions[2] from [2, 5] providing a nice classification of the $(d+1)$-tuples of the extended Delaunay graph $DT(S')$. In particular, let $v(P_0, \ldots, P_d)$ denote the center of the hyperball $C(P_0, \ldots, P_d)$ of $d+1$ sites $P_0, \ldots, P_d \in S$, we have:

$$
\begin{aligned}
\{P_0, \ldots, P_d\} \in DT(S') \quad &\Longleftrightarrow \quad v(P_0, \ldots, P_d) \text{ is a Voronoi point in } VD(S). \\
&\Longleftrightarrow \quad C(P_0, \ldots, P_d) \text{ contains no point of } S \text{ in its interior.} \\
&\Longleftrightarrow \quad \forall_{P' \in S \setminus \{P_0, \ldots P_d\}}\ \mathrm{OUTSIDE}(P_0, \ldots, P_d, P') := \\
&\qquad\qquad \mathrm{sign}\,[\mathrm{VOL}(P_0, \ldots, P_d) * \mathrm{INS}(P_0, \ldots, P_d, P')] \;=\; 1
\end{aligned}
$$

Naturally, an analogous statement can be given for the extended $(d+1)$-tuples. If $\{P_0, \ldots, P_d\}$ and $\{P_0, \ldots, P_{d-1}, \infty\}$ are adjacent $(d+1)$-tuples in $DT(S')$ with $VOL(P_0, \ldots, P_d) > 0$, we have:

$$
\begin{aligned}
\{P_0, \ldots, P_{d-1}, \infty\} \in DT(S') \quad &\Longleftrightarrow \quad P_0, \ldots, P_{d-1} \text{ are the vertices of a } (d-1)\text{-face on} \\
&\qquad\quad \text{the boundary of the convex hull } \partial CH(S). \\
&\Longleftrightarrow \quad \forall_{P' \in S \setminus \{P_0, \ldots P_{d-1}\}}\ \mathrm{OUTSIDE}(P_0, \ldots, P_{d-1}, \infty, P') := \\
&\qquad\qquad \mathrm{sign}\,[\mathrm{VOL}(P_0, \ldots, P_{d-1}, P')] \;=\; 1
\end{aligned}
$$

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[2]These functions VOL and INS (mnemonic for "volume" and "insphere") are defined as follows:

$$
\mathrm{VOL}(P_0, \ldots, P_d) := \begin{vmatrix} 1 & P_{01} & \cdots & P_{0d} \\ \vdots & \vdots & & \vdots \\ 1 & P_{d1} & \cdots & P_{dd} \end{vmatrix}, \quad \mathrm{INS}(P_0, \ldots, P_{d+1}) := \begin{vmatrix} 1 & P_{01} & \cdots & P_{0d} & P_{01}^2 + \cdots + P_{0d}^2 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & P_{d1} & \cdots & P_{dd} & P_{d1}^2 + \cdots + P_{dd}^2 \\ 1 & P_{d+11} & \cdots & P_{d+1d} & P_{d+11}^2 + \cdots + P_{d+1d}^2 \end{vmatrix}
$$

$\mathrm{VOL}(P_0, \ldots, P_d) = 0$ and $\mathrm{INS}(P_0, \ldots, P_{d+1}) = 0$ if the points lie on some common hyperplane or hypersphere, respectively. These properties allow the computation of the topological events in the dynamic case.

Now, we turn over to the dynamic case. For that, we are given a finite set of $n \geq d+2$ continuous trajectory curves in d-dimensional Euclidean space $\mathbb{E}^d$, $S := S(t) := \{P_1(t), \ldots, P_n(t)\}$ under the assumption that the points move without collisions and that there exists an instant t_0 when the set $S(t_0)$ is in general position.

Using the classifications above, we achieve the following compact classification of the elementary topological events of higher dimensional dynamic Voronoi diagrams (compare [2]).

Lemma 1 *Elementary changes of the topological structure $DT(S')$ of the Voronoi diagram $VD(S)$ are characterized by (i,j)-transitions of adjacent $(d+1)$-tuples in $DT(S')$, except in degenerate cases. Thereby, the indices obey the conditions $i+j = d+2$ and $2 \leq i,j \leq d$.*

In other words, non-degenerate topological events happen when $d+2$ neighboring sites of the topological structure $DT(S')$ lie on a common hypersphere (or hyperplane, if ∞ is involved). At this instant, i Delaunay $(d+1)$-tuples which exist shortly before the topological event are replaced by $d-i+2$ $(d+1)$-tuples after the event. Notice that in the planar case, we obtain the classical SWAPs, i.e. $(2,2)$-transitions of neighboring Delaunay triangles (see [11]). In three dimensions, the only possible transition is the $(2,3)$-transition which is depicted in Figure 1 (cf. [1]). Algorithms for maintaining the Voronoi diagram over time can be found in [2, 6]. They use $O(\log n)$ time per event which can be shown to be worst-case optimal [13].

The best known upper bound on the number of topological events of higher dimensional Voronoi diagrams (due to [2]) is given by the following lemma. It is achieved under the following additional *non-periodicity* condition: we make the natural assumption that there exist at most $s \in O(1)$ zeros of the functions INS(...) and VOL(...) (for each combination of the sites) which are computable in constant time each. Indeed, this additional non-periodicity condition can be guaranteed, e.g., in the case of polynomial curves of bounded degree. Notice that this assumption implies that each subset of S' of size $d+2$ generates at most a constant number of topological events and, following that, a trivial $s\binom{n+1}{d+2} \in O(n^{d+2})$ upper bound on the number of topological events. By a Davenport-Schinzel argument, this naive upper bound can be brought down by (roughly) a linear factor (cf. [2] for more details).

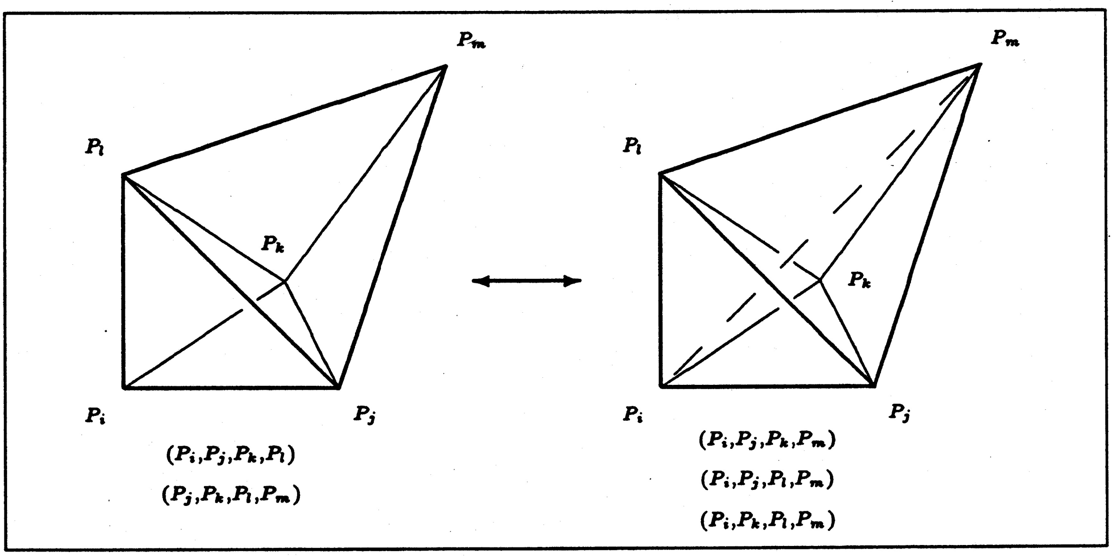

Figure 1: A reversible (2,3)-transition with the active Delaunay $(d+1)$-tuples in $\mathbb{R}^3$.

Lemma 2 *Given a finite set $S(t)$ of n continuous trajectories in d-dimensional Euclidean space $\mathbb{E}^d$, the maximum number of topological events over time is $O(n^d\,\lambda_s(n))$. If only $k \leq n$ points of S are moving (while the remaining $n-k$ stay fixed), this upper bound becomes $O(\alpha_s(k,n))$ where $\alpha_s(k,n) := k\,n^{d-1}\,\lambda_s(n) + (n-k)^d\,\lambda_s(k)$.*

In opposite to that, the known lower worst-case bound is given by the following class of examples (compare [7, 9, 15]). Imagine $n-k$ points fixed such that the corresponding Voronoi diagram has complexity $O((n-k)^{\lceil d/2 \rceil})$ (which is the worst that can happen) and such that the circumspheres of the Delaunay $(d+1)$-tuples can be stuck by a common line.

After that, we make each of the k remaining points after the other pass along this line. Using the classification of the Delaunay $(d+1)$-tuples above, all $O((n-k)^{\lceil d/2 \rceil})$ Delaunay tuples are destroyed during this flow. If we leave sufficient time between these flows, the topological sub-structure of the static points is destroyed only by one point of the moving points each. Therefore every moving point generates $\Omega((n-k)^{\lceil d/2 \rceil})$ topological events.

3 The New Upper Bound

The main topic of this section is to prove a new upper bound on the number of topological events which may appear during the entire flow of the points. We'll see that only $O(\alpha_s(k,n))$ topological events can happen where $\alpha'_s(k,n) := \min\{k^{d+1}(n-k)^{\lceil d/2 \rceil}, \alpha_s(k,n)\}$.

To achieve this, we have to investigate the topological events in more detail. The basic idea behind this approach is a refined counting method. Instead of simply distinguishing between fixed and moving $(d+1)$-tuples (as it was done in [2, 6]) we study the topological events according to the number of moving points which are involved in the event.

Recall that exactly $d+2$ sites of S' are involved in each topological event.[3] Thus, we separately count the number of topological events which are created by exactly i moving points, for $i = 1, \ldots, d+2$. We let t_i denote the number of these special events, respectively. We start with the investigation of those topological events where exactly one moving point is involved while the remaining points are taken from the $n-k$ fixed points. Thereby, we restrict ourselves to non-extended topological events – extended ones (where ∞ is involved) can be handled analogously.

Case 1 : one moving point and $d+1$ fixed points

According to the classification of topological events, the circumcircle of the $d+1$ fixed points involved can not contain any of the $n-k$ fixed points in its interior. Thus, these $d+1$ sites form a Delaunay d-face in the Delaunay graph of the fixed sites.

Now, there are only k moving points and at most $O((n-k)^{\lceil d/2 \rceil})$ such tuples (cf. [3, 9, 15]). Due to our non-periodicity assumption, each of these combinations can only generate $s \in O(1)$ topological events. Thus, we obtain at most

$$t_1 \in O(k\,(n-k)^{\lceil d/2 \rceil})$$

topological events of this type.

Case 2 : i moving points and $d-i+2$ fixed points ($2 \leq i \leq d+1$)

Consider the instant when all i moving and $d-i+2$ fixed points lie on a common hypersphere (due to our classification). As the $d-i+2$ points are fixed, they belong to a $d-i+1$-face of the Delaunay graph of the $n-k$ fixed sites. Using the estimation on the (dual) Voronoi faces given in [3], the number of $d-i+1$-faces is bounded by $O((n-k)^{\min\{i,\lceil d/2 \rceil\}})$.

Now, there exist only $\binom{k}{i}$ possibilities to select the i moving points which generate at most $O((n-k)^{\min\{i,\lceil d/2 \rceil\}})$ topological events, each. Thus, for any fixed i we obtain at most

$$t_i \in O(k^i\,(n-k)^{\min\{i,\lceil d/2 \rceil\}})$$

events.

[3] It should be clear that at least one of them has to move in order to generate some topological event.

Case 3 : $d+2$ moving points and no fixed points

In this case, we apply the earlier result by Albers et al. [2] providing an upper bound of

$$t_{d+2} \in O(k^d \lambda_s(k))$$

topological events of this type.

Finally, summing up these type-i topological events we obtain

$$\begin{aligned}\sum_{i=1}^{d+2} t_i &\leq C \left[\sum_{i=1}^{d+1} k^i (n-k)^{\min\{i,\lceil d/2 \rceil\}} + k^d \lambda_s(k)\right] \\ &\leq C \left[(d+1)\, k^{d+1} (n-k)^{\lceil d/2 \rceil} + k^d \lambda_s(k)\right] \\ &\in O\left(k^{d+1} (n-k)^{\lceil d/2 \rceil} + k^d \lambda_s(k)\right)\end{aligned}$$

for some positive constant C. Notice, that for $k \in O(\sqrt{n})$ (the domain we are interested in) the term $O(k^d \lambda_s(k))$ is dominated by the term $O(k^{d+1} (n-k)^{\lceil d/2 \rceil})$. The following theorem summarizes the new upper bound on the number of topological events.

Theorem 1 *Given a finite set $S(t)$ of k continuously moving points and $n-k$ fixed points in d-dimensional Euclidean space $\mathbb{E}^d$, the maximum number of topological events over time is $O(\alpha'_s(k,n))$ where $\alpha'_s(k,n) := \min\{k^{d+1} (n-k)^{\lceil d/2 \rceil}, \alpha_s(k,n)\}$.*

Thus, if k is considered to be some constant, our counting technique provides an upper bound of $O(n^{\lceil d/2 \rceil})$ topological events instead of $O(n^d)$ with the former approach. This new upper bound matches the lower worst-case bound outlined above in this case. Figure 2 illustrates the known upper and lower worst-case bounds.

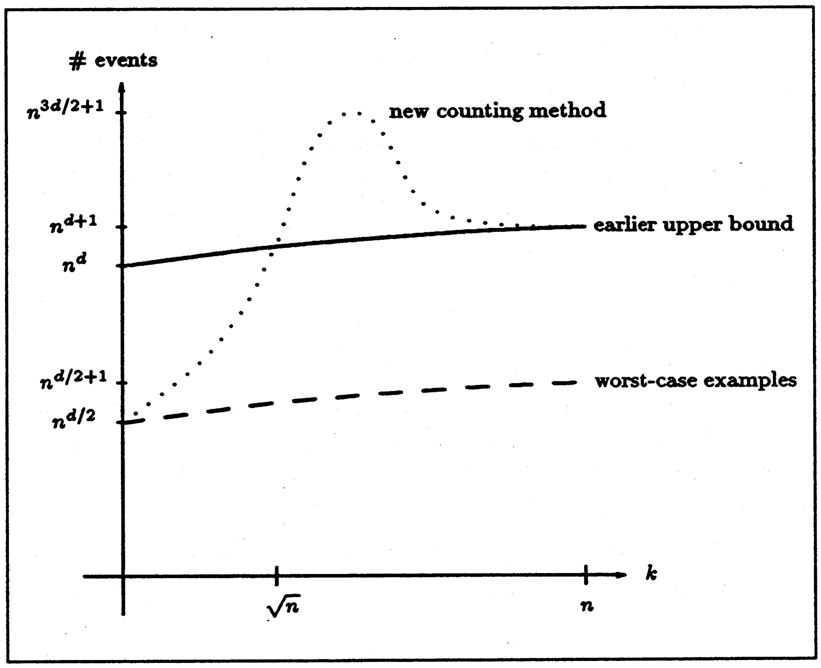

Figure 2: The upper and lower worst-case bounds for dynamic Voronoi diagrams.

4 Conclusions & Acknowledgement

We have presented a new upper bound on the number of topological events which approaches the lower worst-case bound known so far up to a factor of $O(\min\{k^d, (n-k)^{\lfloor d/2 \rfloor}\})$. Future research should further tighten this gap. Nevertheless, this is the first approach with matching upper and lower worst-case bounds for a constant number of moving sites.

Another open problem is the classification of topological events which appear in higher-order higher-dimensional Voronoi diagrams of moving points. Using the technique presented in [2] the analysis of these kinds of events is imaginable. Finally, nearly nothing is known on Voronoi diagrams of moving points under other metrics than the Euclidean one, e.g., L_p-metrics.

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