Systematic local flip rules are generalized Delaunay rules

(Extended Abstract)

Timothy Lambert*

Abstract

A locally optimized triangulation (LOT) of a set of sites can be defined by a flip rule that determines which diagonal of a convex quadrilateral should be included in the triangulation. A flip rule is systematic if there is a unique LOT. It is local if the only new edges added when inserting a new site in the triangulation are adjacent to the new site. I prove that the only systematic local flip rules correspond to generalizations of convex distance function Delaunay triangulations which I call *empty shape triangulations*.

1 Introduction

Definition. In a triangulation, if two adjacent triangles, ABC and ACD, form a convex quadrilateral $ABCD$ it is possible to perform a *flip* and replace the diagonal AC with BD to get the triangles ABD and BCD.

Definition. A *flip rule* is a function $Q(ABCD) \rightarrow \{AC, BD, \textbf{either}\}$ where $ABCD$ is a quadrilateral, which tells us whether the triangulation of $ABCD$ should include the diagonal AC or the diagonal BD. **either**(Q) is the set of quadrilaterals (regarded as points in $\mathbf{R}^8$) that flip rule Q returns **either** for, and similarly for $AC(Q)$ and $BD(Q)$. We require $AC(Q)$ and $BD(Q)$ to be open sets and **either**(Q) the boundary between them.

Definition. A *locally optimized triangulation* (LOT) with respect to a flip rule Q is one where the flip rule would not change any diagonal of any convex quadrilateral formed by a pair of adjacent triangles.

Definition. A flip rule is *systematic* if there is a unique LOT, no matter what the site set is.

Definition. A flip rule is *local* when the only edges added to a LOT when a site is added to the triangulation and flips performed to construct a new LOT are those adjacent to the new site.

If a triangulation is locally Delaunay then it is globally Delaunay [18], so the flip rule that selects the Delaunay triangulation of the quadrilateral (DT) is systematic and local.

In this paper I prove that DT is the only flip rule invariant under translation and rotation of the sites which is systematic and local.

Definition. Given a collection of shapes such that every triangle has a unique circumscribing[1] shape, an *empty shape triangulation* of a set of a sites is a set of triangles such that the circumscribing shape contains no site in its interior.

If the shapes are the set of homothets of a convex set, then the empty shape triangulation is the convex distance function (cdf) Delaunay triangulation [5]. I prove that a homothetic [2] systematic local flip rule is an empty shape triangulation where the shapes are homothets of a *set of* convex sets.

1.1 Background

Triangulating site sets is a very important problem in computational geometry; there are far too many applications in computational geometry and other fields to mention here. (See the surveys [3, 6, 1])

There are many different possible triangulations of a set of sites. Which one is optimal will depend on the application. For example:

- If the triangulation is to be used as finite element mesh we wish to avoid ill-conditioned equations. This means that we do not want triangles with angles close to 180° [2].
- If we are using the triangulation to linearly interpolate functions with a bounded second derivative, then the error is minimized by minimizing the maximum circumradius of any triangle [17].
- If the triangulation represents a three-dimensional surface which is to be rendered on

*School of Computer Science and Engineering, University of New South Wales, PO Box 1, Kensington NSW 2033, Australia. T.Lambert@unsw.edu.au

[1] circumscribing means that the triangle vertices lie on the boundary of the shape

[2] one that is invariant under scalings and translations

a raster display, then we want to avoid triangles less than one pixel wide as these can cause undesirable artifacts [7].

Many other alternative definitions of optimality have been proposed. See [3, 13] for surveys.

Lawson [14] introduced the idea of using flips to maximize the minimum angle to improve a triangulation. By elementary geometry we can prove that this is the same flip rule as DT. Because DT is systematic the very simple flip algorithm (repeatedly apply the flip rule) will converge to the Delaunay triangulation, so the Delaunay triangulation maximizes the minimum angle over all triangulations—it is a globally optimized triangulation (GOT).

Because DT is local, the incremental algorithm [15, 11] will make at most $O(n)$ changes to the triangulation when adding a new site, giving a worst case of $O(n^2)$ time to compute the Delaunay triangulation. Guibas, Knuth and Sharir [10] show that the average[3] number of sites adjacent to a new site is $O(1)$, so the incremental algorithm will make an average of $O(n)$ changes to the triangulation.

The divide-and-conquer algorithm [15] for the Delaunay triangulation also depends on the local property, since it guarantees that the new edges added when merging two triangulations separated by a line are just those that cross the line and are ordered by their intersection with that line.

Nielson and Franke [16] claim the flip rule "choose the triangulation that minimizes the maximum angle in both triangles" is systematic. It is possible to find a simple counterexample to their claim, but this leads to the question: Which flip rules are systematic and local?

2 Main results

Definition. The lines making up the sides of a triangle ABC divide the plane into seven regions (see figure 1). We will denote by $AB\overline{C}$ the region that is adjacent to the points A and B, but not C. The regions are open sets, their union is the plane except for the lines AB, BC, and AC.

Definition. $Q^+(ABC)$ is the set of points in the plane for which the flip rule Q would not choose the triangle ABC. That is, if $D \in A\overline{B}C$ then $D \in Q^+(ABC)$ iff $Q(ABCD) = BD$. We can similarly define $Q^+(ABC)$ in the regions $\overline{A}BC$ and $AB\overline{C}$. In the remaining regions, the quadrilateral is not convex, so the flip rule must choose the interior diagonal. This means that ABC is included in $Q^+(ABC)$ and $\overline{A}\overline{B}C$, $\overline{A}B\overline{C}$, $A\overline{B}\overline{C}$ are

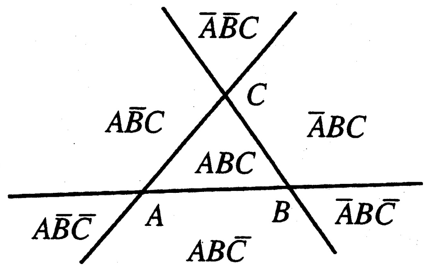

Figure 1: Regions around a triangle

excluded. $Q^0(ABC)$ is the boundary of $Q^+(ABC)$, and $Q^-(ABC) = \mathbf{R}^2 - (Q^+(ABC) + Q^0(ABC))$

Definition. Let Q^0 be the set of curves $Q^0(ABC)$ for all possible triangles ABC.

Definition. We say that Q is *circumscribing* if, given any triangle, there is exactly one curve in Q^0 circumscribing that triangle.

Definition. The *flip graph* of a set of sites: The nodes consist of all possible triangulations of that set. Two nodes are connected by an edge if one can be transformed into the other by a single flip. A flip rule gives a direction to each edge.

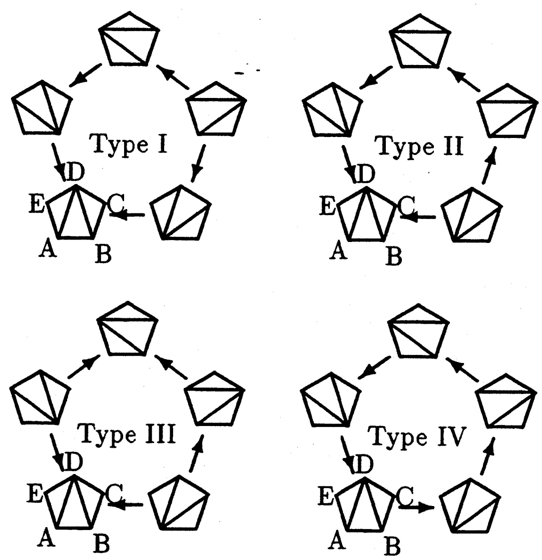

Figure 2: Possible directed flip graphs

There are only four possible different directed flip graphs for a set of sites that forms a convex pentagon (see figure 2).

- Type IV: No sink. No LOT exists. The flip rule is not systematic.
- Type III: Two sinks. LOT is not unique. The flip rule is not systematic.

[3] taken over all insertion orders

- Type II: One sink. The flip rule is not local: The (unique) LOT of $ACDE$ is ACE, CDE since EC is preferred to DA but ADE, ABD, BD is the LOT of $ABCDE$. The edge AD is a new edge in this triangulation although it is not adjacent to B.

- Type I: One sink. A systematic local flip rule will always have this form.

2.1 Systematic local rules are circumscribing

Lemma 1. *If Q is a systematic local rule and $C \in Q^0(ABE) \cap \overline{A}BE$ then $Q^+(ABE) = Q^+(ACE)$ in the region $\overline{A}CE - \overline{AB}C$ (the shaded region in figure 3).*

PROOF. If $Q^+(ABE) \neq Q^+(ACE)$ then there is a point $D \in Q^+(ABE) \setminus Q^+(ACE)$ or a point $D \in Q^+(ACE) \setminus Q^+(ABE)$ (see figure 3). Note that $ABCDE$ is strictly convex. Let's consider the first

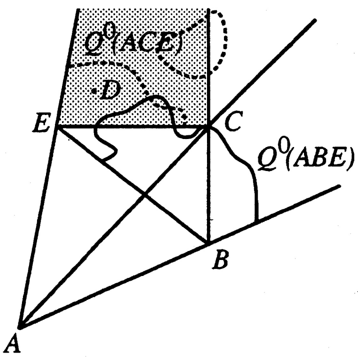

Figure 3: $Q^0(ABE)$ and $Q^0(ACE)$

case. Either $D \in Q^-(ACE)$ or $D \in Q^0(ACE)$. If $D \in Q^0(ACE)$ then because $Q^+(ABE)$ is open and $Q^0(ACE)$ is the boundary of $Q^-(ACE)$ we can find a new $D' \in Q^+(ABE) \cap Q^-(ACE)$.

- $ABCD'E$ is strictly convex
- $Q(ABD'E) = AD'$
- $Q(ACD'E) = CE$
- $Q(ABCE) =$ **either**

Now, because $AD'(Q)$ and $CE(Q)$ are open sets we find a ball around $ABCE$ such that for all $A'B'C'E'$ in that ball $Q(A'B'D'E') = A'D'$, $Q(A'C'D'E') = C'E'$ and $A'B'C'D'E'$ is convex. And since $ABCE$ is on the boundary between $AC(Q)$ and $BE(Q)$ we can find $A'B'C'E'$ in that ball such that:

- $A'B'C'D'E'$ is strictly convex
- $Q(A'B'C'E') = B'E'$
- $Q(A'B'D'E') = A'D'$
- $Q(A'C'D'E') = C'E'$

There is no way that we can pick the directions of the two remaining edges in the flip graph for $A'B'C'D'E'$ so that it is type I (see figure 4). This contradicts Q being local and systematic.

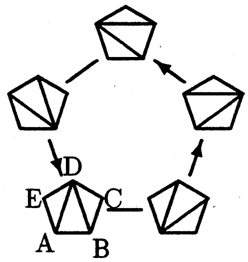

Figure 4: Not a type I flip graph

If $D \in Q^+(ACE) \setminus Q^+(ABE)$, the proof is the same, except that we find $A'B'C'E'$ such that $Q(A'B'C'E') = A'C'$. This leads to a flip graph that is figure 4 with all the arrows reversed. This still can't be type I. □

Lemma 2. *If Q is a systematic local rule and $C \in Q^0(ABE) \cap \overline{A}BE$ then $Q^+(ABE) = Q^+(BCE)$ in the region $\overline{A}CE - \overline{AB}C$ (the shaded region in figure 3).*

PROOF. Omitted. □

Lemma 3. *If Q is a systematic local rule and $C \in Q^0(ABE) \cap \overline{A}BE$ then $Q^+(ABE) = Q^+(ACE) = \emptyset$ in the region $\overline{AB}C$*

PROOF. Omitted □

Lemma 4. *If Q is a systematic local rule and $D \in Q^0(ABE) \cap \overline{A}BE$ then $(BDE \cap ABD) \subset Q^+(ABE)$*

PROOF. Omitted □

Lemma 5. *If Q is a systematic local rule and $D \in Q^0(ABC)$ then $Q^+(ABC) = Q^+(BCD) = Q^+(ACD) = Q^+(ABD)$.*

PROOF. D cannot be in $ABC \cup A\overline{BC} \cup \overline{A}B\overline{C} \cup \overline{AB}C$. Relabel the sites if necessary, so that $D \in A\overline{B}C$ (see figure 5). Note that $C \in Q^0(ABD)$, $B \in Q^0(ACD)$ and $A \in Q^0(BCD)$. We need to prove the result in the regions $\overline{A}CD$ and $ACD \cap BCD$ (shaded in figure 5). The rest will follow by symmetry.

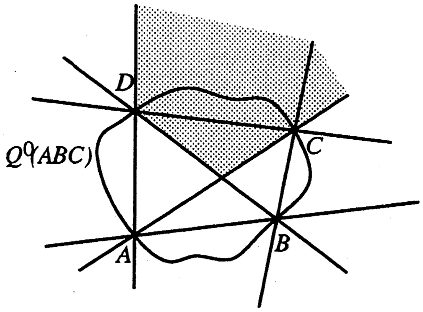

Figure 5:

- In the region $\overline{ACD} \cap \overline{BCD}$ applying lemma 1 with ABE replaced by ABC shows that $Q^+(ABC) = Q^+(BCD)$ and with ABE replaced by ABD shows that $Q^+(ABD) = Q^+(ACD)$. Applying lemma 2 with ABE replaced by ABC shows that $Q^+(ABC) = Q^+(ACD)$.

- In the region $\overline{ABC}$, lemma 3 shows that $Q^+(ABC) = Q^+(ABD) = \emptyset$. Because Q is a flip rule $Q^+(ABC) = \emptyset$, and $Q^+(BCD) = \emptyset$ because $\overline{ABC} \subset \overline{BCD}$.

- in the region $BCD \cap ACD = I$ applying lemma 4 with ABE replaced by ABC shows that $I \subset Q^+(ABC)$ and with ABE replaced by ABD shows that $I \subset Q^+(ABD)$. $I \subset Q^+(BCD)$ because $I \subset BCD$ and $I \subset Q^+(ACD)$ because $I \subset ACD$.

Finally we note that we have left out the lines AB, BC and AC in the proof, but there is only one way to complete $Q^+(ABC)$ onto these lines. □

Theorem 1. *If Q is a systematic local rule and $D, E, F \in Q^0(ABC)$ then $Q^+(ABC) = Q^+(DEF)$. (That is, Q is circumscribing.)*

Proof. By lemma 4, $Q^0(ABC) = Q^0(ABD) = Q^0(ADE) = Q^0(DEF)$. □

Theorem 2. *If Q is a systematic local rule then $Q^+(ABC)$ is convex.*

Proof. If $Q^+(ABC)$ is not convex, then there are points $D, E \in Q^+(ABC)$ and $F \notin Q^+(ABC)$ such that F lies on the line segment $\overline{DE}$ (see figure 6). If $F \in Q^0(ABC)$ then we can find new DEF such that $D, E \in Q^+(ABC)$ and $F \in Q^-(ABC)$. The segment $\overline{DF}$ goes from inside $Q^+(ABC)$ to outside, so let D' be a point where it intersects $Q^0(ABC)$, and E' a point where $\overline{EF}$ intersects $Q^0(ABC)$. Let $G \in Q^0(ABC) \setminus DE$. (If we can't

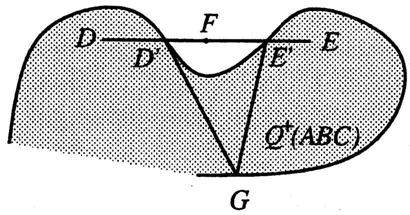

Figure 6: $Q^+(ABC)$ is convex

find such a G then $Q^0(ABC) = DE$, which contradicts $F \in Q^-(ABC)$.) Since F is on the edge of $D'E'G$ and $F \in Q^-(ABC) = Q^-(D'E'G)$ (by theorem 1) we can find an $F' \in D'E'G \cap Q^-(D'E'G)$ which contradicts Q being a flip rule. □

2.2 Circumscribing rules are systematic and local

The proof that DT is systematic and local [11] relies on the following geometric fact: If two circles share a common chord, then on each side of the chord the interior of one circle is a subset of the interior of the other circle.

If Q is circumscribing then the same fact is true, provided we replace "circle" with "curve from $\mathcal{Q}^0$". Consequently the same proof proves that circumscribing rules are systematic and local.

Furthermore, just as for the Delaunay triangulation we have the "empty circle property"—the circumcircle of each Delaunay triangle contains no other site, for $GOT(Q)$, Q systematic and local, we have the "empty shape property"—the circumscribing curve for each triangle $GOT(Q)$ contains no other site.

2.3 The only rotation and translation invariant systematic local flip rule is DT.

Definition. A directed line l is a *support line* of a set S iff l contains a boundary point (a *support point*) of S and S is contained in the closed half-plane to the left of l.

Definition. The two asymptotes of an unbounded curve are the limits of the support lines as the support point goes to infinity.

Theorem 3. *The only rotation and translation invariant systematic local flip rule is DT.*

Proof. Let Q be such a rule and $K \in \mathcal{Q}^0$.

Case 1 K is bounded. Fujiwara [8] and Bol [4] have shown that if K is a compact convex set which is not a disc, then it is possible to find an infinite number of congruent copies K' of K such that K and K' have at least four points in common on their boundaries. It follows from Theorem 1 that $K = K'$. Hence K has an infinite number of symmetries and must be a disc.

Case 2 K is unbounded. Let A_1 and A_2 be the asymptotes to K, P their point of intersection, and α the angle between them. Rotate K by $\alpha/2$ about P, translate by d in direction A_1 and by d in direction $A_2 + \alpha/2$ to get K' (figure 7). By making d sufficiently large we can ensure

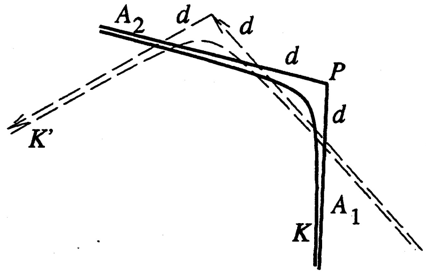

Figure 7: K and K' intersect three times

that A_2 is a chord of K'. Then the boundaries of K and K' intersect three times and by theorem 1 $K = K'$, i.e. $\alpha = 0$ and K is a half-plane, which we can regard as infinitely large disc.

Hence $Q^0(ABC)$ is a circle for any three points A, B, and C and by Theorem 1 this circle is the circumcircle of ABC and $Q = DT$. □

2.4 The only systematic local homothetic flip rules are generalized Delaunay rules.

At this point it might seem that a systematic local homothetic rule Q must necessarily be that of a Delaunay triangulation based on the convex distance function induced by the "circle" $Q^0(ABC)$ [5], but this is only true if Q^0 contains only homothets of one shape.

Suppose $Q^0(ABC)$ is a square (see figure 8). This is what you get in the l_∞ metric. If D is in the shaded region in figure 8 then it is not possible to draw a scaled, translated copy of the square $Q^0(ABC)$ through the sites ADB. $Q^0(ADB)$ must be a different shape, for example, the bottom right corner of a circle joined to an upwards ray and a leftwards ray.

In general, the Q^0 curves will be homthets of a collection of shapes. We can complete the example with three more copies of $Q^0(ABD)$ rotated through 90°, 180° and 270°.

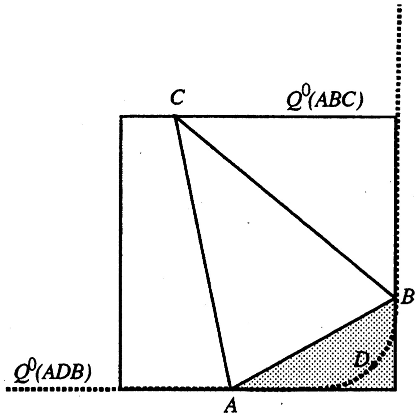

Figure 8: Two different Q^0 curves

Theorem 4. *Let Q be a homothetic systematic local flip rule. Then $K \in Q^0$ is either strictly convex (contains no straight line segments in its boundary) or a cone (boundary is two rays).*

PROOF. Omitted □

Theorem 5. *If there is a finite number of shapes in Q^0 then there is one bounded shape with all the other shapes "rounding" off the corners of this shape.*

PROOF. Omitted □

If Q^0 contains an infinite number of shapes then the result is similar, except that it is possible to have an infinite number of cones to fill in the space between two cones.

How does this relate to cdf Delaunay triangulation? We can define the Euclidean Delaunay triangulation by the "empty circle property"—the triangles of the Euclidean Delaunay triangulation are just those whose circumcircles contain no other site. Similarly, the cdf Delaunay triangulation can be defined by the "empty shape property"—the triangles are those whose circumscribing shapes are empty, where the shape is just the unit circle for the cdf. The triangulation described in theorem 5 is a further generalization to a set of shapes instead of a single shape.

If the cdf is not strictly convex then theorem 4 tells us that the corresponding flip rule is not local and systematic. The problem here is that in this case triangles with a side parallel to a line segment on the boundary of the shape have more than one circumscribing shape and the Delaunay triangulation is not unambiguously defined. To resolve this ambiguity one particular circumscribing shape must be chosen (e.g. the bottom leftmost one [12]). This is effectively treating the flat part of the boundary as being very slightly curved, and the conditions of theorem 4 hold.

If the cdf has corners then the conditions of theorem 5 are violated because there is no shape that rounds the corners of the cdf. The problem here is that triangles with two sides that are support lines at the same corner of the cdf cannot be circumscribed and the cdf Delaunay triangulation does not completely triangulate the convex hull of the input sites. In this case we can add shapes to round the corners of the convex distance function and the cdf Delaunay triangulation is just a subset of the generalized one. This leads to simple new algorithms (including a sweepline algorithm) for cdf Delaunay triangulation (implementation details are in [13]) and constrained cdf Delaunay triangulation.

Finally we note that we can interpret our generalized cdf Delaunay triangulation as duals of Voronoi diagrams in the surreal [9] Cartesian plane, where the distance function is smooth (no corners) and strictly convex.

3 Conclusion

Locally optimized triangulations are simple to define and compute, while the systematic and local properties are important and useful properties for a flip rule to possess.

I have shown that the only translation and rotation invariant flip rule is the rule DT. This gives some more reasons to support the use of the Delaunay triangulation as the most natural and useful triangulation of a set of points.

I have shown that the only homothetic flip rules correspond to empty shape Delaunay triangulations, which generalize convex distance function Delaunay triangulations. Any Delaunay triangulation algorithm can be modified to produce empty shape Delaunay triangulations and consequently convex distance function Delaunay triangulations.

References

[1] F. Aurenhammer. Voronoi diagrams—a survey of a fundamental geometric data structure. *Computing Surveys*, 23(3):345–405, 1991.

[2] I. Babuška and A. Aziz. On the angle condition in the finite element method. *SIAM J. Numer. Analysis*, 13:214–227, 1976.

[3] M. Bern and D. Eppstein. Mesh generation and optimal triangulation. In D.-Z. Du and F. K. Hwang, editors, *Computing in Euclidean Geometry*, volume 1 of *Lecture Notes Series on Computing*, pages 23–90. World Scientific, Singapore, 1992.

[4] G. Bol. Zur kinematischen Ordnung ebener Jorden-Kurven. *Abh. Math. SEm. Univ. Hamburg*, 11:394–408, 1936.

[5] Robert L. (Scot) Drysdale, III. A practical algorithm for computing the Delaunay triangulation for convex distance functions. In *Proc. 1st ACM-SIAM Sympos. Discrete Algorithms*, pages 159–168, 1990.

[6] Leila De Floriani. Surface representation based on triangular grids. *The Visual Computer*, 3:27–50, 1987.

[7] James D. Foley, Andries van Dam, Steven K. Feiner, and John F. Hughes. *Computer Graphics: Principles and Practice*. Addison-Wesley, Reading, MA, 1990.

[8] M. Fujiwara. Ein Satz über konvexe geschlossene Kurven. *Sci. Repts. Tôhoku Univ.*, 9:289–294, 1920.

[9] Harry Gonshor. *An introduction to the theory of surreal numbers*, volume 110 of *London Mathematical Society lecture note series*. Cambridge University Press, Cambridge, 1986.

[10] L. J. Guibas, D. E. Knuth, and M. Sharir. Randomized incremental construction of Delaunay and Voronoi diagrams. *Algorithmica*, 7:381–413, 1992.

[11] L. J. Guibas and J. Stolfi. Primitives for the manipulation of general subdivisions and the computation of Voronoi diagrams. *ACM Transactions on Graphics*, 4:74–123, 1985.

[12] R. Klein. *Concrete and Abstract Voronoi Diagrams*, volume 400 of *Lecture Notes in Computer Science*. Springer-Verlag, 1989.

[13] T. Lambert. Local optimization of triangulations. manuscript, 1993.

[14] C. L. Lawson. Generation of a triangular grid with application to contour plotting. Technical Memorandum 299, California Institute of Technology Jet Propulsion Laboratory, 1972.

[15] D. T. Lee and B. J. Schachter. Two algorithms for constructing the Delaunay triangulation. *International journal of Computer and Information Sciences*, 9:219–242, 1980.

[16] G. M. Nielson and R. Franke. Surface construction based upon triangulations. In R. Barnhill and W. Boehm, editors, *Surfaces in Computer-Aided Geometric Design*, pages 163–177. North Holland, 1983.

[17] Stephen M. Omohundro. The Delaunay triangulation and function learning. Technical Report TR-90-001, International Computer Science Institute, Berkeley, CA, 1990.

[18] R. Sibson. Locally equiangular triangulations. *The Computer Journal*, 21(3):243–245, 1978.