ON EQUIANGULARITY OF DELAUNAY DIAGRAMS IN EVERY DIMENSION

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INTRODUCTION.-

Given a set S of sites in the d-dimensional Euclidian space E, the Voronoi diagram Vor(S) of S is such that every region of Vor(S) is the set of points of E strictly closer to one of the sites of S than to all the others. Vor(S) admits a dual diagram Del(S) called the Delaunay diagram of S.

These diagrams have been studied extensively and a large number of their properties are known in the plane [Aur91, Ede87, PS85]. Sibson [Sib77] was the first to observe that every locally equiangular triangulation in the plane is a completion of the Delaunay diagram. H. Edelsbrunner [Ede87] pointed out that every triangulation which maximizes lexicographically the increasing sequence of angles that appear in the triangles of a triangulation is also a completion of the Delaunay diagram. D. Mount and A. Saalfeld [MS88] showed how to build efficiently these triangulations. Moreover, V.T. Rajan [Raj91] has extended some metric properties of the Delaunay triangulation in the plane to every dimension.

We generalize here the notion of equiangularity to the partitions in inscribable polyhedra and we show that the new equiangularity characterizes the Delaunay diagrams in the plane. This can not be extended to higher dimensions.

Furthermore, we introduce a dual notion, the coequiangularity, and show that it characterizes the Delaunay diagrams in every dimension.

1.- Delaunay diagrams and recessive facets.

We give here a characterization of Delaunay diagrams in every dimension, using a property of their facets.

A diagram in a d-dimensional Euclidian space E, is a partition of E in a finite number of 0-faces or vertices, of 1-faces or edges, of 2-faces, ..., of d-1-faces or facets and of d-faces or regions.

Definition 1.1.- A diagram D of E such that every bounded region of D is a d-dimensional inscribable convex polyhedron is called an inscribable diagram.

S being a set of sites of E, let Dins(S) be the set of inscribable diagrams that admit the sites of S as vertices. Dins(S) contains, in particular, every triangulation of S.

Definition 1.2.- A facet f of a diagram D of Dins(S) is said to be recessive if the two regions p and q of D, having f as a common facet, are bounded and such that $p \subset \omega(q)$, where $\omega(q)$ is the open ball circumscribed to q. $p \subset \omega(q)$ is equivalent to $q \subset \omega(p)$.

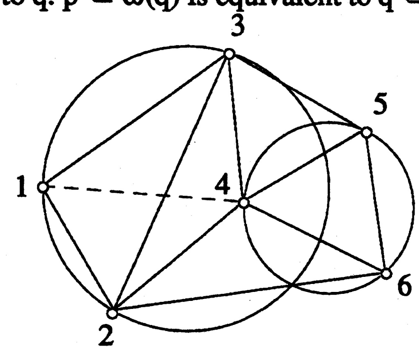

Fig 1

On the planar example from figure 1, the only recessive facet is the edge 23.
If this edge 23 is removed and replaced by the edge 14, we obtain the Delaunay diagram.

In dimension 2, this notion of recessive facet is to be compared with the notion of illegal edge introduced by H. Edelsbrunner [Ede87]. An illegal edge is recessive but the reciprocal is wrong.

Theorem 1.3.- For every finite set S of sites in a d-dimensional Euclidian space E, the Delaunay diagram Del(S) is the only diagram of Dins(S) without any recessive facet.

Proof.- (i).- The Delaunay diagram of a set S of sites of E is a partition of E such that all its regions are inscribable and convex. Furthermore for every region r of Del(S) and for every site s of $S\setminus\delta(r)$, $s \notin \overline{\omega}(p)$ ($\delta(r)$ is the boundary of r and $\overline{\omega}(p)=\omega(p) \cup \delta(\omega(p))$). Thus, a Delaunay diagram has no recessive facet.

(ii).- Let D be a diagram of Dins(S) different from Del(S). Hence, there exist a region p of D and a site s of $S\setminus\delta(p)$ such that $s \in \overline{\omega}(p)$.

There exists $z \in p$ such that the open straight-line segment sz does not pass through any site of S and does not cut any k-face of D, for every $k \in \{1, \ldots, d-2\}$. We now prove the existence of a recessive facet by a recurrence on the number of facets of D cut by sz for such a point z.

(ii.1).- Since $s \notin \overline{p}$ and $z \in p$, there exists at least one facet f of p cut by sz.

(ii.2).- If sz cuts exactly one facet f of D, then f is a facet of p and s is a vertex of the region q having f as a common facet with p. Since $s \in \overline{\omega}(p)$ and q is inscribable, $q \subset \omega(p)$ and hence f is recessive.

(ii.3).- Let us suppose the following recurrence hypothesis : "For every point z' of a region p' of D, such that sz' cuts h facets and $s \in \overline{\omega}(p')$, sz' cuts at least one recessive facet" and let us show that the same property is also true for h+1 facets.

Suppose now that sz cuts h+1 facets and let f be the facet of p cut by sz and q the region of D having f as a common facet with p.

If $q \subset \omega(p)$, the facet f is recessive and the result is true. Otherwise, since $s \in \overline{\omega}(p)$ and s and z are on both sides of f, $s \in \overline{\omega}(q)$. By replacing z by a point $z' \in q \cap sz$, sz' cuts h facets of D and, according to the recurrence hypothesis, sz' cuts at least one recessive facet and sz also cuts this facet.□

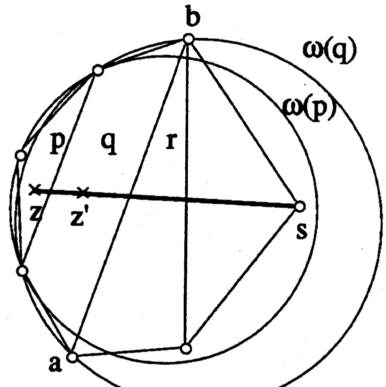

Fig 2

In the example of figure 2, the regions p, q and r are such that, ω(p) and ω(q) contain s whereas ω(r) does not contain s. Thus q is included in $\overline{\omega}(r)$ and the edge ab is recessive.

2.- Delaunay diagrams and equiangularity.-

We generalize the notion of equiangularity and we prove that Del(S) is the only equiangular diagram of Dins(S) in the plane.

Definition 2.1.- Let D be a diagram of Dins(S), p a bounded region of D and f a facet of p. Let c be the center of ω(p), $s \in \delta(f) \cap S$, F the half-space delimited by the hyperplane of f and that does not contain p and cu the half-line with c as endpoint, orthogonal to the hyperplane of f and such that $F \cap cu$ is unbounded.

The geometric angle (cs,cu) defined by the half-lines cs and cu is said to be associated to the facet f relatively to the region p. This angle is denoted by α(f,p).

In the particular case where the region p is unbounded, the hyperplane of facet f is the limit of a sphere with center at infinity and we pose α(f,p) = 0. This particular case only occurs in part 3.

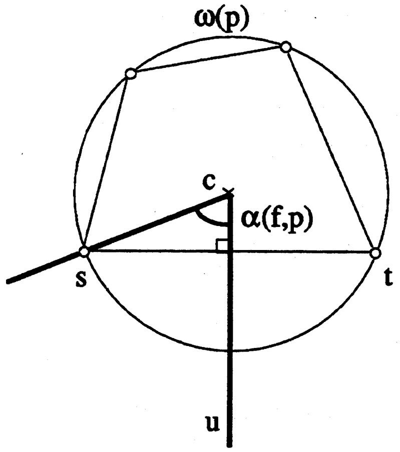

Fig 3.a

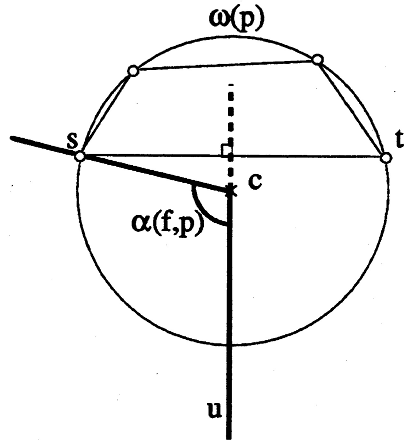

Fig 3.b

In the planar examples of figure 3, the facet f is the edge st of an inscribable region p. The angle $\alpha(f,p)$ corresponds to the half-angle under which the edge st is seen from the center c of $\omega(p)$ in figure 3.a and is supplementary to this half-angle in figure 3.b.

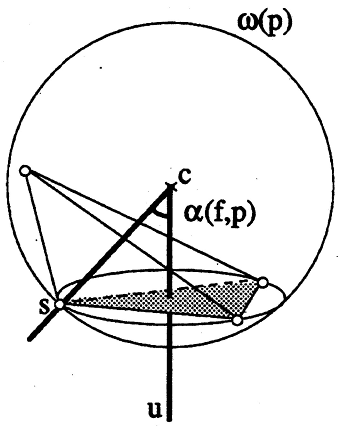

Fig 4.a

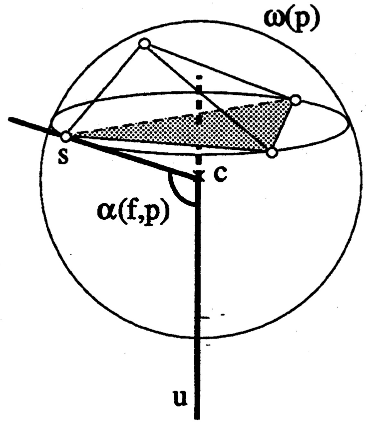

Fig 4.b

In the 3-dimensional examples of figure 4, f is the (grey) facet from a tetrahedron on the boundary of a region p. Let H be the cone with the center c of $\omega(p)$ as vertex and that leans upon the circle circumscribed to f. The angle $\alpha(f,p)$ is the half-angle at the vertex of H for the tetrahedron in figure 4.a and is supplementary to it in figure 4.b.

Definition 2.2.- (i).- For every diagram D of Dins(S), let $A(D) = (\alpha_1(D), \alpha_2(D), \dots \alpha_m(D))$ be the increasing sequence of the non-zero angles associated to all the facets of D and $|A(D)| = m$ the length of the sequence A(D).

(ii).- P and Q being two diagrams of Dins(S), let the lexicographical order relation be such that $A(P) < A(Q)$ if

- either there exists j such that, $\forall\, i < j$, $\alpha_i(P) = \alpha_i(Q)$ and $\alpha_j(P) < \alpha_j(Q)$.
- or A(Q) is an initial subsequence of A(P), i.e $|A(Q)| < |A(P)|$ and, $\forall\, i \in \{1, \dots, |A(Q)|\}$, $\alpha_i(Q) = \alpha_i(P)$

(iii).- A diagram P of Dins(S) is said to be equiangular if, $\forall\, Q \in$ Dins(S), $A(Q) \le A(P)$.

In the case where S is a set of coplanar sites such that there are not more than 3 cocircular sites, the equiangularity defined here corresponds to the notion of globally equiangularity introduced by H. Edelsbrunner [Ede87]. In fact, for every triangle t with vertices a,b,c, $(ca,cb) = \alpha(ab,t)$.

Remark 2.3.- It results from the previous definition that if A(Q) is a subsequence of A(P), i.e. A(Q) is obtained by removing some elements from A(P), then $A(P) < A(Q)$. In fact, either A(Q) is an initial subsequence of A(P) or there exist j and $k > j$ such that, $\forall\, i < j$, $\alpha_i(Q) = \alpha_i(P)$ and $\alpha_j(Q) = \alpha_k(P) > \alpha_j(P)$.

Theorem 2.4.- For every finite set S of coplanar sites, the Delaunay diagram Del(S) is the only equiangular diagram of Dins(S).

Proof.- Let D be a diagram of Dins(S) which is not Delaunay. By theorem 1.3, there exist two regions p and q of D adjacent to a same facet such that $p \subset \omega(q)$. Let $L = \delta(p) \cap S$, $R = \delta(q) \cap S$ and $L \cap R = \{a,b\}$. Let H be the restriction of D to $L \cup R$.

Case 1.- If the sites of $L \cup R$ are cocircular then every edge of $Del(L \cup R)$ is on the boundary of $conv(L \cup R)$, the convex hull of $L \cup R$. Consequently, $A(Del(L \cup R))$ is a subsequence of A(H) and, by remark 2.3, $A(H) < A(Del(L \cup R))$.

Case 2.- If the sites of $L \cup R$ are not cocircular and if the smallest angle $\alpha_1(Del(L \cup R))$ of $A(Del(L \cup R))$ is an angle $\alpha(ss',r)$ with $ss' \subset \delta(conv(L \cup R))$, then we can suppose, within a permutation of L and R, that $s, s' \in L$. Since $r \subset \omega(p)$, we have $\alpha(ss',p) < \alpha(ss',r)$. It follows that $\alpha_1(H) \leq \alpha(ss',p) < \alpha_1(Del(L \cup R))$ and $A(H) < A(Del(L \cup R))$ [see figure 5.a].

Case 3.- If the sites of $L \cup R$ are not cocircular and if $\alpha_1(Del(L \cup R))$ is an angle $\alpha(gd,r)$ with $g \in L\backslash R$ and $d \in R\backslash L$, then we can suppose, within a permutation of a and b, that a and r are on the same side of the straight-line gd.

Since $a, b, g \in \delta(\omega(p))$, we have $(ag,ab) \geq \alpha_1(H)$. Moreover, since $a \notin \omega(r)$, $(ag,ad) \leq \alpha(gd,r)$. Thus $(ag,ab) < (ag,ad)$ implies $\alpha_1(H) < \alpha_1(Del(L \cup R))$ and $A(H) < A(Del(L \cup R))$ [see figure 5.b].

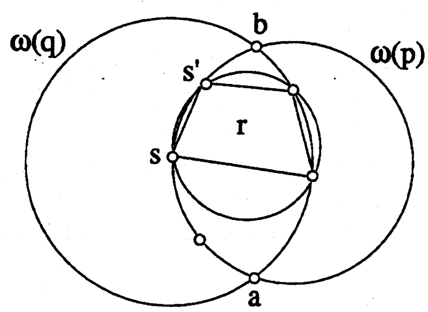

Fig 5.a

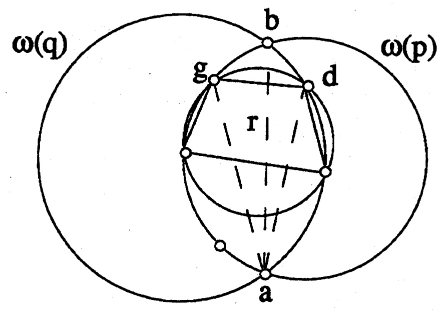

Fig 5.b

If Q is the diagram obtained by replacing H by $Del(L \cup R)$ in D, we have $A(D) < A(Q)$ and D is not equiangular.

Thus Del(S) is the only diagram of Dins(S) that is equiangular.□

Remark 2.5.- (i).- The uniqueness of the equiangular diagram established by theorem 2.4 is not true for globally equiangular triangulations as figure 6 shows.

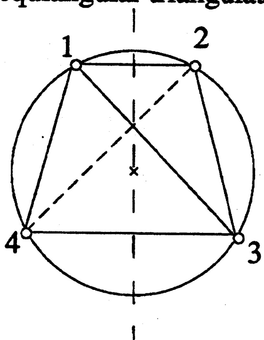

Fig 6

Let S = {1,2,3,4} be a set of 4 cocircular sites such that {1,2} and {3,4} admit the same perpendicular bisector. Hence, there exist exactly two triangulations T1 and T2 of S and A(T1) = A(T2).

(ii).- The introduction of the zero angles in A(D) does not modify the result of theorem 2.4. In fact, the zero angles would form an initial subsequence Z(D) of A(D) and, $\forall$ $D \in Dins(S)$, $|Z(D)| = |Z(Del(S))|$.

Remark 2.6.- The generalization of theorem 2.4 in dimension $d > 2$ is wrong. The following example proves this in case $d = 3$.

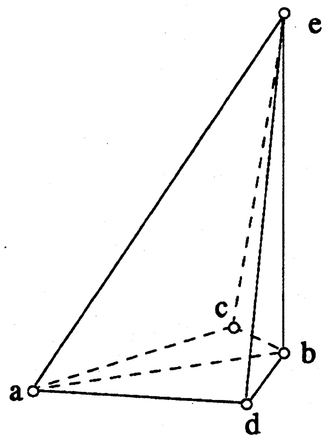

Fig 7.a

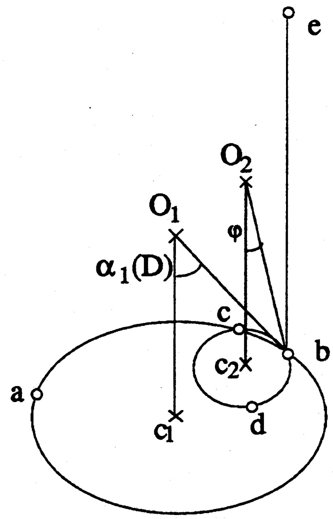

Fig 7.b

Let S = {a,b,c,d,e} where a,b,c,d are coplanar, ab is the perpendicular bisector of c and d, a and b are on both sides of the straight-line cd and be is orthogonal to the plane abcd.

If a is out of the circle passing through b,c,d, figure 7.a gives the only diagram D of Dins(S) that is not Delaunay. The angles associated to the facets abc and abd relatively to the polyhedra abce and abde respectively, are equal and if be is big enough, these angles are the smallest angles of A(D). Let $\alpha_1(D)$ be the angle associated to the facet abc.

Let O_1 and O_2 be the centers of the spheres passing respectively through a,b,c,e and b,c,d,e and let c_1 and c_2 be the centers of the circles abc and bcd. Since be is orthogonal to the plane abcd, $|O_1c_1| = |O_2c_2| = |be| / 2$ and thus, as $|c_2b| < |c_1b|$, the angle φ associated to the facet bcd relatively to the polyhedron bcde of Del(S) is such that $\varphi < \alpha_1(D)$.

Hence Del(S) is not equiangular.

3.- Delaunay diagrams and coequiangularity.-

We introduce the notion of coequiangularity and we prove that Del(S) is the only coequiangular diagram of Dins(S) in every dimension.

Definition 3.1.- (i).- Let S be a set of sites of E and D a diagram of Dins(S). For every facet f of D common to two regions p and q, the angle $\beta(f) = \alpha(f,p) + \alpha(f,q)$ is said to be coassociated to the facet f.

(ii).- For every diagram D of Dins(S), let $\beta(D)$ be the greatest angle coassociated to the facets of D.

A diagram P of Dins(S) is said to be coequiangular if, $\forall$ Q $\in$ Dins(S), $\beta(Q) \geq \beta(P)$.

Theorem 3.2.- For every finite set S of sites in the d-dimensional space, the Delaunay diagram Del(S) is the only coequiangular diagram of Dins(S).

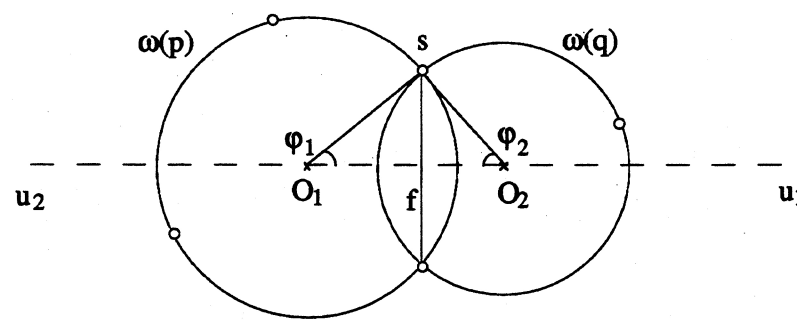

Fig 8.a

Proof.- (i).- If f is a facet of Del(S) common to two bounded regions p and q then f is not recessive, from theorem 1.3.

Let s be a vertex of f, O_1 and O_2 the respective centers of $\omega(p)$ and $\omega(q)$ and O_1u_1, O_2u_2 two half-lines such that $\varphi_1 = \alpha(f,p) = (O_1s, O_1u_1)$ and $\varphi_2 = \alpha(f,q) = (O_2s, O_2u_2)$ [see figure 8.a].

As $\omega(q) \not\supset p$, $O_1u_1 \cap O_2u_2 \neq \emptyset$ et $\varphi_1 + \varphi_2 + (sO_1, sO_2) = \pi$. It results that $\beta(f) = \varphi_1 + \varphi_2 < \pi$.

(ii).- If D is a diagram of Dins(S) different from Del(S) then, by theorem 1.3, D admits at least one recessive facet f. With the notations of (i) we thus have, $p \subset \omega(q)$ [see figure 8.b].

- if the vertices of p and q are cocircular then $O_1 = O_2$ and $\beta(f) = \varphi_1 + \varphi_2 = \pi$.
- if the vertices of p and q are not cocircular then $O_1u_1 \cap O_2u_2 = \emptyset$ and $(\pi - \varphi_1) + (\pi - \varphi_2) + (sO_1, sO_2) = \pi$. It follows that $\beta(f) = \varphi_1 + \varphi_2 = \pi + (sO_1, sO_2) > \pi$.

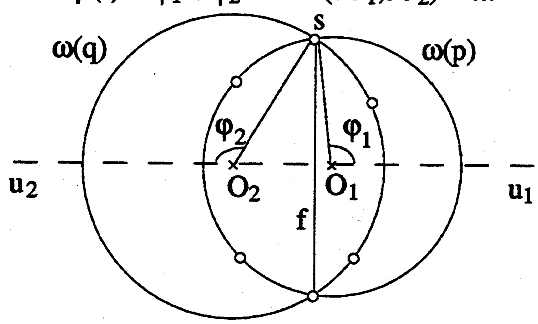

Fig 8.b

(iii).- If f is a facet of $\delta(\text{conv}(S))$ then, $\forall$ D $\in$ Dins(S), at least one of the angles associated to f is zero and $\beta(f) < \pi$.

From (i), (ii) and (iii) it results that if D is a Delaunay diagram then for every facet f of D, $\beta(f) < \pi$ and if D is not Delaunay then there exists at least one facet f of D such that $\beta(f) \geq \pi$. Hence, Del(S) is the only coequiangular diagram of Dins(S).□

CONCLUSION.-

A new definition of the angles associated to a facet, has allowed us to extend the notion of equiangularity to the diagrams whose regions are inscribable convex polyhedra. Moreover we have characterized the Delaunay diagram in the plane by using these angles.

By adding up the angles facet per facet, we have defined a new notion, the coequiangularity, that is dual of equiangularity since it minimizes the maximum angle.

Unlike equiangularity, the property of coequiangularity of Delaunay diagrams is valid in every dimension.

Equiangularity and coequiangularity are two properties which express differently the regularity of Delaunay diagrams. Now the question is to know if this regularity can be expressed by other angular relations.

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