Finding the Largest m-dimensional Circle in a k-dimensional Box

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Abstract

Everett and Whitesides solved the problem of finding the largest (2-dimensional) circle in a 3-dimensional box. In this paper we give a two-way generalization by proving that the largest m-dimensional circle in k-dimensional box ($m \le k$) of size $2a_1 \times 2a_2 \times \ldots \times 2a_k$ where $0 < a_1 \le a_2 \le \ldots \le a_k$ has the radius $\sqrt{\dfrac{a_1^2 + a_2^2 + \ldots + a_{k-s}^2}{m-s}}$, where s is the minimal number among 0,1,2,..., m-1 for which $(m-s-1)a_{k-s}^2 \le a_1^2 + a_2^2 + \ldots + a_{k-s-1}^2$ is satisfied. All possible positions of the center of the largest circle are given and for m=k-1, m=1 and m=k the largest circles are found to be unique (for each possible position of the center), up to symmetries.

1. Introduction

Everett and Whitesides [EW, EW1] have shown how to find all the largest 2-dimensional circles inside a 3-dimensional box. In particular, they show that the radius of the largest circle in a box of dimension $2a \le 2b \le 2c$ is equal to $\sqrt{a^2+b^2}$ for $c^2 \ge a^2 + b^2$, and $\sqrt{\dfrac{a^2+b^2+c^2}{2}}$ otherwise. We generalize the solution in both circle and box dimensions.

A k-dimensional box B of size $2a_1 \times 2a_2 \times \ldots \times 2a_k$ is a set of points $(x_1, x_2, \ldots, x_k)$ in k-dimensional Euclidean space R^k such that $-a_i \le x_i \le a_i$, for each i, $1 \le i \le k$. Without loss of generality we may assume that $0 < a_1 \le a_2 \le \ldots \le a_k$. The box is bounded by 2k facets which belong to some hyperplanes. The i-th front and i-th back facet belong to hyperplanes that have equations $x_i=a_i$ and $x_i=-a_i$, respectively, i=1,2,...,k. The box is centered at the origin, i.e. point (0,0,...,0). We refer to m-dimensional circle and k-dimensional box simply as circle and box, respectively.

A set of m vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ in R^k is linearly independent if $\alpha_1\mathbf{v}_1+\alpha_2\mathbf{v}_2+\ldots+\alpha_m\mathbf{v}_m=0$ is possible only when $\alpha_1=\alpha_2=\ldots=\alpha_m=0$, where $\alpha_1, \alpha_2, \ldots, \alpha_m$ are real numbers. Clearly $m \le k$. Note that in the sequel vectors are denoted in bold face letters.

A m-dimensional flat M in R^k is a set of points $\mathbf{c}+\alpha_1\mathbf{v}_1+\alpha_2\mathbf{v}_2+\ldots+\alpha_m\mathbf{v}_m$, where $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ is a linearly independent fixed set of vectors from E^k, $\mathbf{c}=(c_1, c_2, \ldots, c_k)$ is a point from E^k and $\alpha_1, \alpha_2, \ldots, \alpha_m$ are variable real numbers. In short, this would be denoted $M=\{\mathbf{c}+\alpha_1\mathbf{v}_1+\alpha_2\mathbf{v}_2+\ldots+\alpha_m\mathbf{v}_m\}$. A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ is called orthonormal if and only if it satisfies the following properties: $|\mathbf{v}_i|=1$, i=1,2,...,m, and $\mathbf{v}_i\mathbf{v}_j=0$ for $1 \le i \ne j \le m$. Recall that $\mathbf{uw}=(u_1,u_2,\ldots,u_k)(w_1,w_2,\ldots,w_k)=u_1w_1+u_2w_2+\ldots+u_kw_k$ is a dot product while $|\mathbf{v}_i|$ is the vector norm, defined by $|\mathbf{v}_i|^2=\mathbf{v}_i\mathbf{v}_i$. Let $\mathbf{v}_i=(v_{i1}, v_{i2}, \ldots, v_{ik})$, $1 \le i \le m$. Then $|\mathbf{v}_i|^2=v_{i1}^2+v_{i2}^2+\ldots+v_{ik}^2=1$. It follows from standard linear algebra (cf. [SB]) that it is always possible to choose an orthonormal set $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ in above representation of M, called orthonormal basis of M, and in the sequel we make such assumption.

The distance between two points $\mathbf{u}$ and $\mathbf{w}$ in E^k is $|\mathbf{u}-\mathbf{w}|$. The distance between a point $\mathbf{u}$ and a flat M is the minimal distance between $\mathbf{u}$ and a point from M.

A m-dimensional circle C of radius r and center $\mathbf{c}=(c_1,c_2,\ldots,c_k)$ in E^k is a set of points in a m-dimensional flat which are at distance no more than r from a fixed point $\mathbf{c}$. Therefore if a point $(x_1,x_2,\ldots,x_k)$ belongs to m-dimensional circle C of radius r and center $\mathbf{c}$ then $(x_1-c_1)^2+(x_2-c_2)^2+\ldots+(x_k-c_k)^2 \le r^2$.

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In this paper we determine the radius of the largest circle inside given box by the formula given in the abstract. Consider few special cases of the formula. For m=1 it gives s=0 and the radius $\sqrt{a_1^2 + a_2^2 + \ldots + a_k^2}$ of the largest 1-dimensional circle, i.e. line segment, corresponds to the main diagonal of the box. For m=k we obtain s=k-1 and radius a_1 is the minimal box size. The result of [EW, EW1] is the special case of the formula for m=2 and k=3.

The next section contains the proof the the formula for the radius of the largest circle inside given box. The proof is not a straightforward generalization of [EW, EW1]. Here we give an illustration by an informal interpretation of lemmas in the next section for the case m=2 and k=3 (which is the case that can be visualized).

Let M be the plane containing the largest circle C, and $c=(c_1, c_2, c_3)$ be its center. Lemma 1 will state that the distance from c to the intersection of M with i-th facet is proportional to $|a_i-c_i|$. Lemma 2 proves that the largest circle can be translated such that the center moves to the origin. Assuming that $a_1 \le a_2 \le a_3$, Lemma 3 shows that if C is tangent to the j-th back and front facets than it is tangent to i-th back and front facets for any i<j. Consequently Lemma 4 shows that C can be tangent to either only 1-st, or 1-st and 2-nd, or all back and front facets. In Lemma 5 the radius is proved to be $\sqrt{\frac{a_1^2 + a_2^2 + a_3^2}{2}}$ in case when C is tangent to all facets of the box. Lemma 6 shows that if C is not tangent to the 3-rd back and front facets than its radius is the same as the radius of the largest 1-dimensional circle (line segment) in the box $2a_1 \times 2a_2$, i.e. the diagonal of the rectangle. This solves the problem for m=2 and k=3.

2. Finding the radius of the largest circle

Let $M_i(a_i)$ be the intersection of flat M and hyperplane $x_i=a_i$, i.e. $M_i(a_i)= \{c+\alpha_1 v_1+\alpha_2 v_2+\ldots + \alpha_m v_m\} \cap \{x_i=a_i\}$. $M_i(-a_i)$ is defined similarly. In general case, $M_i(a_i)$ and $M_i(-a_i)$ are (m-1)-dimensional flats.

Lemma 1. The distance from point **c** to flat $M_i(a_i)$ is $\dfrac{|a_i-c_i|}{\sqrt{\sum_{j=1}^{m} v_{ji}^2}}$.

Proof. The distance between **c** and any point $c+\alpha_1 v_1+\alpha_2 v_2+\ldots + \alpha_m v_m$ from M is $|c+\alpha_1 v_1+\alpha_2 v_2+\ldots + \alpha_m v_m - c|=|\alpha_1 v_1+\alpha_2 v_2+\ldots + \alpha_m v_m|$. Let $d_i=\alpha_1 v_1+\alpha_2 v_2+\ldots + \alpha_m v_m$. Then $|d_i|^2=d_i d_i=(\alpha_1 v_1+\alpha_2 v_2+\ldots + \alpha_m v_m)(\alpha_1 v_1+\alpha_2 v_2+\ldots + \alpha_m v_m)=\alpha_1^2 + \alpha_2^2 + \ldots + \alpha_m^2$, which follows easily from the properties of an orthogonal set of unit vectors. Our objective is to minimize the last function subject to constraint that the point $c+\alpha_1 v_1+\alpha_2 v_2+\ldots + \alpha_m v_m$ belongs to hyperplane $\{x_i=a_i\}$. Without loss of generality assume i=1 and $v_{m1}\neq 0$. Then the first coordinate in point $c+\alpha_1 v_1+\alpha_2 v_2+\ldots + \alpha_m v_m$ is a_1, i.e. $c_1+ \alpha_1 v_{11}+ \alpha_2 v_{21}+ \ldots + \alpha_m v_{m1} =a_1$. Thus $\alpha_m=\frac{1}{v_{m1}}(a_1- c_1- \alpha_1 v_{11}- \alpha_2 v_{21}-\ldots - \alpha_{m-1}v_{m-1,1})$. Our task now is to minimize the function $|d_1|^2= f(\alpha_1,\ldots,\alpha_{m-1})= \alpha_1^2 + \alpha_2^2 + \ldots + \alpha_{m-1}^2 + \frac{1}{v_{m1}^2}(a_1-c_1-\alpha_1 v_{11}-\alpha_2 v_{21}-\ldots -\alpha_{m-1}v_{m-1,1})^2$. The function is obviously continuous and has all partial derivatives. It achieves extreme value when all partial derivatives $\frac{\partial f}{\partial \alpha_i}=0$ for all i=1,2,...,m-1. Therefore $2\alpha_i + \frac{2}{v_{m1}^2}(a_1-c_1-\alpha_1 v_{11}-\alpha_2 v_{21}-\ldots -\alpha_{m-1}v_{m-1,1})(-v_{i1})=0$ or $\frac{\alpha_i}{v_{i1}}= \frac{1}{v_{m1}^2}(a_1-c_1-\alpha_1 v_{11}-\alpha_2 v_{21}-\ldots -\alpha_{m-1}v_{m-1,1})$. The right hand side does not depend on i and thus $\frac{\alpha_1}{v_{11}}=\frac{\alpha_2}{v_{21}}= \ldots = \frac{\alpha_{m-1}}{v_{m-1\,1}}$. From this it follows that $\alpha_j=\frac{v_{j1}}{v_{11}}\alpha_1$ for j=2,3,..., m-1. On the other hand $\alpha_m=\frac{1}{v_{m1}}(a_1- c_1- \alpha_1 v_{11}- \alpha_2 v_{21}-\ldots - \alpha_{m-1}v_{m-1,1})= \frac{v_{m1}}{v_{11}}\alpha_1$. Substituting this in above equation gives

$$\frac{\alpha_1}{v_{11}} v_{m1}^2 = a_1 - c_1 - \alpha_1 v_{11} - \frac{v_{21}}{v_{11}} \alpha_1 v_{21} - \dots - \frac{v_{m-1\,1}}{v_{11}} \alpha_1 v_{m-1,1} \text{ or}$$

$$\alpha_1 = \frac{(a_1-c_1)v_{11}}{\sum_{j=1}^{m} v_{j1}^2}. \qquad \text{Thus } \alpha_t = \frac{(a_1-c_1)v_{t1}}{\sum_{j=1}^{m} v_{j1}^2} \text{ for } t=1,2,\dots,m.$$

When obtained values are applied to function $|d_i|^2 = \alpha_1^2 + \alpha_2^2 + \dots + \alpha_m^2$ above, the distance is obtained as Lemma indicates.♦

Lemma 2. If m-dimensional circle C with radius r and center **c** is completely inside a k-dimensional box B centered at the origin then the translated circle with radius r and center at the origin is also inside B.

Proof. Let $\mathbf{c'}_1=(-c_1,0,0,\dots,0)$. Consider translation of circle C with radius r and center **c** for vector $\mathbf{c'}_1$. The circle C will be translated to circle C''_1 with center $\mathbf{c''}_1=(0,c_2,c_3,\dots,c_k)$ and radius r. Distances of points **c** and $\mathbf{c''}_1$ from their respective flat $M_i(a_i)$ are, according to Lemma 1, the same for $i>1$. These distances also remain unchanged with respect to their respective flats $M_i(-a_i)$. Therefore if circle **c** is between i-th front and back facets then circle $\mathbf{c''}_1$ is also between i-th front and back facet for $i>1$. Consider now the distances to $M_1(a_1)$ and $M_1(-a_1)$. According to Lemma 1, these distances for point **c** are

$$df = \frac{|a_i - c_i|}{\sqrt{\sum_{j=1}^{m} v_{ji}^2}} \quad \text{and} \quad db = \frac{|-a_i - c_i|}{\sqrt{\sum_{j=1}^{m} v_{ji}^2}}.$$

Similarly, the distances from $\mathbf{c''}_1$ to its corresponding flats $M_1(a_1)$ and $M_1(-a_1)$ are

$$df1 = \frac{|a_i|}{\sqrt{\sum_{j=1}^{m} v_{ji}^2}} \quad \text{and} \quad db1 = \frac{|-a_i|}{\sqrt{\sum_{j=1}^{m} v_{ji}^2}}.$$

Clearly $-a_1 \le c_1 \le a_1$ and thus

$$df+db = df1+db1 = \frac{2a_i}{\sqrt{\sum_{j=1}^{m} v_{ji}^2}}.$$

Since $df1=db1$ it follows that $df+db=2df1$. Therefore $\min(df,db) \le df1$. In other words, the translation does not reduce the distance of the center of circle to closer of 1-st back or front facets (here the distances are measured within the flat that contains the circle). Since circle centered at c is between 1-st front and back facets, it means that the circle of the same radius centered at $\mathbf{c''}_1$ is also between 1-st front and back facets. In conjuction to similar result obtained for other facets, we conclude that circle centered at $\mathbf{c''}_1$ is inside box B.

We now repeat the same analysis, assigning $\mathbf{c''}_1$ the role of **c** and considering the translation for vector $\mathbf{c'}_2=(0,-c_2,0,0,\dots,0)$. After this second translation circle C moves to position where its center is at the point $\mathbf{c''}_2=(0,0,c_3,c_4,\dots,c_k)$. Analogous arguments confirm that the circle remains inside box B. After similar translations performed for vectors $\mathbf{c'}_i=(0,0,\dots,0,-c_i,0,\dots,0)$ (non-zero element is at i-th position), the center of circle is moved to the origin and circle remains inside box B.♦

We may therefore consider only the circles centered at the origin in order to find the largest one that is entirely inside given box. Such assumption is made in the rest of the section. As another consequence of Lemma 1, it follows that circles centered at the origin either are tangent to both i-th back and front facets or are not tangent to any of them, for each $i=1,2,\dots,k$. Here a facet and a circle are tangent to each other if they have exactly one common point. This is equivalent to the center of circle being at distance r from the intersection of the facet and the flat containing circle, where r is the radius of the circle.

Let r^* be the radius of the largest circle inscribed inside given box. Obviously if a largest circle is not tangent to the i-th back and front facets then any largest inscribed circle of box B'_i of size $2a_1 \times 2a_2 \times \dots \times 2a_{i-1} \times 2a'_i \times 2a_{i+1} \times$

... x $2a_k$ also has radius r* for any $a'_i > a_i$, including the case $a'_i=\infty$. For simplicity, any such box will be denoted as $B'_i= 2a_1 \times 2a_2 \times ... \times 2a_{i-1} \times \infty \times 2a_{i+1} \times ... \times 2a_k$. It includes the case of infinite size i-th dimension.

The other obvious property is that if the largest inscribed circle of a box of size $2a_1 \times 2a_2 \times ... \times 2a_k$ has radius r* then the largest inscribed circle of the box of size $2b_1 \times 2b_2 \times ... \times 2b_k$ also has radius r*, where $b_1, b_2, ..., b_k$ is any permutation of $a_1, a_2, ..., a_k$.

Lemma 3. If a largest inscribed circle C' of radius r* is tangent to the j-th back and front facets of box B but is not tangent to the i-th back and front facets where j>i and $a_j \geq a_i$ then C' is a largest inscribed circle of box B'_j and is not tangent to any of the i-th and j-th back and front facets of B'_j.

Proof. If C' is not tangent to the i-th back and front facets then C' is the largest inscribed circle for box B'_i. Therefore any largest inscribed circle inside box B" of size $2a_1 \times 2a_2 \times ... \times 2a_{i-1} \times 2a_j \times 2a_{i+1} \times ... \times 2a_{j-1} \times \infty \times 2a_{j+1} \times ... \times 2a_k$ obtained by permuting i-th and j-th size component of B'_i also has radius r*. Clearly box B'_j is completely inside box B" since each size component of B'_j is ≤ each size component of B". Since on the other hand B is completely inside B'_j it follows that circle C' with radius r* is inside B'_j and is a largest inscribed circle of box B'_j.♦

Lemma 4. There exist t (1≤t≤k) such that the largest inscribed circle inside box B is tangent to the i-th back and front facets for i=1,2,...,t and its radius r* is the same as the radius of the largest inscribed circle inside any box of size $2a_1 \times 2a_2 \times ... \times 2a_t \times 2a'_{t+1} \times ... \times 2a'_k$ where $a'_{t+1}>a_{t+1}, ..., a'_k>a_k$ are arbitrary size components.

Proof. We may denote the size of later box by $2a_1 \times 2a_2 \times ... \times 2a_t \times \infty \times ... \times \infty$. Let t be such that a largest circle C' is tangent to the i-th back and front facets for i=1,2,...,t but C' is not tangent to the (t+1)-th back and front facets. Then from previous lemma it follows that C' is the largest inscribed circle of any B'_j for j>t, and therefore for box $2a_1 \times 2a_2 \times ... \times 2a_t \times \infty \times ... \times \infty$.♦

Consider now the case t=k which corresponds to circle being tangent to all facets of box B.

Lemma 5. If the largest circle C inside box B is tangent to all facets of B then the radius r* of C is equal to

$$\sqrt{\frac{\sum_{i=1}^{k} a_i^2}{m}}.$$

Proof. If C is tangent to all facets of B then all distances from origin, which is the center of the circle, to a facet of B (within the flat containing circle) are equal to r*, the radius of C.

Let $T_i=\sum_{j=1}^{m} v_{ji}^2$. According to Lemma 1, it follows that $\frac{a_1^2}{T_1}=\frac{a_2^2}{T_2}=...=\frac{a_k^2}{T_k}$. Hence $T_i= T_1\frac{a_i^2}{a_1^2}$ for i=2,3,..., k.

Therefore $\sum_{i=1}^{k} T_i=\frac{T_1}{a_1^2}\sum_{i=1}^{k} a_i^2$.

From $\sum_{i=1}^{k} T_i= \sum_{i=1}^{k}\sum_{j=1}^{m} v_{ji}^2= \sum_{j=1}^{m}\sum_{i=1}^{k} v_{ji}^2= \sum_{j=1}^{m} 1=m$ it follows that $r^{*2}=\frac{a_1^2}{T_1}=\frac{\sum_{i=1}^{k} a_i^2}{m}$.♦

The result of the last lemma correspond to the case s=0 of the major result of the paper. It will serve as the basis of an inductive proof of the formula. The next lemma provides the induction step for the final result.

Lemma 6. The largest (m'-1)-dimensional circle in (k'-1)-dimensional box B' of size $2a_1 \times 2a_2 \times ... \times 2a_{k'-1}$ and the largest m'-dimensional circle in k'-dimensional box B of size $2a_1 \times 2a_2 \times ... \times 2a_{k'-1} \times \infty$ have the same radius.

Proof. Let C be an (m'-1)-dimensional circle of radius r in $E^{k'-1}$, centered at the origin. C is a set of points $X'=(x_1, x_2, ..., x_{k'-1})$ which belong to a (m'-1)-dimensional flat and satisfy the property $x_1^2 + x_2^2 + ... + x_{k'-1}^2 \le r^2$.

Map each such point X' to the point $X=(x_1, x_2, ..., x_{k'-1}, x_{k'})$ from $E^{k'}$ such that $x_1^2 + x_2^2 + ... + x_{k'-1}^2 + x_{k'}^2 = r^2$.

Obviously there are one (for $x_{k'}=0$) or two (otherwise) points X for a given X'. All points X obtained by mapping of all points from C define the border of an m'-dimensional circle $C^{(1)}$ which has the same radius r. The circle $C^{(1)}$ is uniquely determined by C. We refer to so obtained circle $C^{(1)}$ as the 1-st expanded circle of C. The inverse mapping can be similarly defined and obviously the correspondence is one-to-one.

Suppose now that C is the largest (m'-1)-dimensional circle with radius r inside (k'-1)-dimensional box B' of size $2a_1 \times 2a_2 \times ... \times 2a_{k'-1}$. Then for each point $X'=(x_1, x_2, ..., x_{k'-1})$ from C the following property is valid: $|x_i| \le |a_i|$ for i=1,2,...,k'-1. Clearly $C^{(1)}$ is between k'-th back and front facets, since B has arbitrarily large k'-th dimension ($a_{k'}=\infty$). Therefore $|x_i| \le |a_i|$ for i=1,2,...,k' and the 1-st expanded m'-dimensional circle $C^{(1)}$ is inside box B of size $2a_1 \times 2a_2 \times ... \times 2a_{k'-1} \times \infty$. Suppose that there exist a circle C' inside B with radius r'>r. By a similar argument C' is 1-st expanded circle of a (m'-1)-dimensional circle that is inside B' and has radius >r. This contradicts the choice of C. Therefore $C^{(1)}$ is the largest inscribed circle in B. In analogous way one can prove that if $C^{(1)}$ is the largest circle in B then C is the largest circle (of one less dimension) in B'.♦

Define now $C^{(i)}$ to be 1-st expanded circle of $C^{(i-1)}$. Then the following lemma follows directly from Lemmas 5 and 6 by applying induction.

Lemma 7. Let s be such that the largest inscribed circle inside box B of size $2a_1 \times 2a_2 \times ... \times 2a_k$ is tangent to the i-th back and front facets for i=1, 2, ..., k-s and is not tangent to the i-th back and front facets for i=k-s+1, k-s+2,..., k. Then the radius of the largest inscribed circle inside B is $\sqrt{\dfrac{a_1^2 + a_2^2 + ... + a_{k-s}^2}{m-s}}$.♦

We are now ready to prove the major result of this paper.

Theorem 1. The largest m-dimensional circle in k-dimensional box ($m \le k$) of size $2a_1 \times 2a_2 \times ... \times 2a_k$ where $0 < a_1 \le a_2 \le ... \le a_k$ has the radius $\sqrt{\dfrac{a_1^2 + a_2^2 + ... + a_{k-s}^2}{m-s}}$, where s is the minimal number among 0,1,2,..., m-1 for which $(m-s-1)a_{k-s}^2 \le a_1^2 + a_2^2 + ... + a_{k-s-1}^2$ is satisfied.

Proof. The radius r of the largest circle C is determined in Lemma 7. Lemma 7 defines s such that C is tangent to the i-th back and front facets of B for i=1,2,..., k-s and is not tangent for i>k-s. Then from the construction of 1-st extended circles it follows that $a_i > r$ for i>k-s since (0,0,...,0) maps to (0,0,...,0, ±r); i.e. box dimension is not tangent to the largest circle iff the appropriate box size is greater than the radius. From

$$a_{k-s+1} > \sqrt{\frac{a_1^2 + a_2^2 + ... + a_{k-s}^2}{m-s}},$$ it follows that $(m-s)a_{k-s+1}^2 > a_1^2 + a_2^2 + ... + a_{k-s}^2$.

Then for j<s we obtain $(m-j)a_{k-j+1}^2 = (m-s)a_{k-j+1}^2 + (s-j)a_{k-j+1}^2 > (m-s)a_{k-s+1}^2 + a_{k-s+1}^2 + a_{k-s+2}^2 + ... + a_{k-j}^2 > a_1^2 + a_2^2 + ... + a_{k-j}^2$. Therefore s is the minimal number among 0,1,2,..., m-1 for which $(m-s-1)a_{k-s}^2 \le a_1^2 + a_2^2 + ... + a_{k-s-1}^2$ is satisfied.♦

3. Finding all largest inscribed circles

It would be of interest to determine all largest inscribed circles in addition to finding their diameter. In this section we solve the problem in several cases.

For m=1 obviously there are 2^{k-1} main diagonals, each being a largest line segment.

In case m=k the center $c=(c_1,c_2, ..., c_k)$ of circle satisfies $c_1=0$ and $-a_i+a_1 \le c_i \le a_i-a_1$ for $2 \le i \le k$.

In general case, the center $c=(c_1,c_2, ..., c_k)$ of circle satisfies $c_i=0$ for $i=1,2,...,t$ and $-a_i+r \le c_i \le a_i-r$ for $t < i \le k$, where r is the radius of the largest circle and t is determined by Lemma 4. More precisely, any point **c** satisfying these properties is center of a largest circle.

While the problem of determining all center position is solved by the last observation, the problem of finding all largest circles with given center (say, at the origin) remains to be investigated. We partially answer this question. The following theorem gives an answer for m=k-1. This is a one-way generalization of the results presented in [EW, EW1].

Theorem 2. Let $\mathbf{n}=(n_1,n_2,..., n_k)$ be a (uniquely determined) unit normal vector to (k-1)-dimensional flat containing the largest circle. If the largest (k-1)-dimensional circle inside k-dimensional box is tangent to all facets then $n_i^2 = 1 - \dfrac{ma_i^2}{a_1^2 + \dots + a_k^2}$ and there are 2^{k-1} different largest inscribed circles, all centered at the origin.

Proof. Let A be the matrix having $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_{k-1}}$ and **n** as rows. These vectors form an orthonormal basis of E^k. Therefore $AA^T=I$ where I is identity square matrix and A^T is the transpose of A. For square matrices A and B, from AB=I it follows that BA=I (see exercize 3 section 2.7 of [SB]). Therefore $A^TA=I$. From this it follows that $v_{1i}^2 + v_{2i}^2 + \dots + v_{k-1\ i}^2 + n_i^2 = 1$ for $i=1,2,..., k$. Thus $\dfrac{a_i^2}{r^2} + n_i^2 = 1$; therefore n_i is determined as Theorem 2 states.

There exist another proof of the same result. It is well known that $\cos \beta_i = n_i$ where β_i is the angle formed by **n** and i-th back and front facets. On the other hand, $\sin \beta_i = a_i/r = \sqrt{1-n_i^2} = \sqrt{1-\cos^2(\beta_i)}$, therefore n_i^2 is determined.

There are two different values for n_i which satisfy the same formula. This gives a total of 2^k normal vectors. However, **n** and **-n** define the same hyperplane, and the number of different circles is 2^{k-1}.♦

If the largest circle is not tangent to all facets in Theorem 2 then similar conclusion can be obtained using Lemma 6 and above observation on the position of the center of circle.

Conclusion

In case of general m and k, the number of restrictions on the orthonormal basis appears to be insufficient to uniquely (up to symmetries) determine the flat containing the largest inscribed circle. It is not even known whether the number of solutions is finite. This problem remains to be investigated.

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