On the Minimal Number of Volume Intersection Operations Necessary for Reconstructing a 3D Object

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Abstract

*Volume intersection is an object reconstruction technique which supplies a boundary volume **R** of an unknown 3D object **O**. It consists in intersecting the volumes obtained by back–projecting a number of 2D silhouettes of **O**. Several algorithms have been presented for computing **R** under different hypotheses relative to viewpoints, type of projection (parallel or perspective), representation of the reconstructed object. Not any concave object **O** is exactly reconstructable by volume intersection: the closest approximation of **O** that can be obtained with this approach is its* visual hull*, a recently introduced geometric entity. Only objects coincident with their visual hulls are, in theory, exactly reconstructable(*e–reco).

*In practice, to exactly reconstruct an object we must also face computational problems. This paper addresses the problem of finding the theoretical minimal number of intersections necessary for exactly reconstructing an object **O**, or its visual hull if **O** is not e–reco. Among other results, we have found that to reconstruct polyhedra with a bounded number* n *of faces may take an unbounded number of intersections. In the case of viewpoints also lying inside the convex hull of an e–reco polyhedron, we show that* $O(n^5)$ *intersections are sufficient, and give an algorithm for finding the viewpoints.*

I. Introduction

Reconstructing 3D shapes from 2D images is an important area of research in computer vision. Possible applications range from the representation of a robot's workspace to the construction of models of human organs. The technique known as *volume intersection* [1–17]constructs a representation of a 3D object **O** starting from a set of silhouettes S_i of **O**. With the word silhouette we indicate the region of a 2D image of an object **O** which contains the projections of the visible points of the object.

The volume intersection technique(Fig.1) recovers a volumetric description $\mathbf{R}_n$ of the object by intersecting the solid regions of space C_i within which each silhouette S_i constrains the object to lie:

$$\mathbf{R}_n = \bigcap_{i=1}^{n} C_i$$

The reconstructed volume $\mathbf{R}_n$ is therefore a bounding volume which more or less closely approximates **O**. For

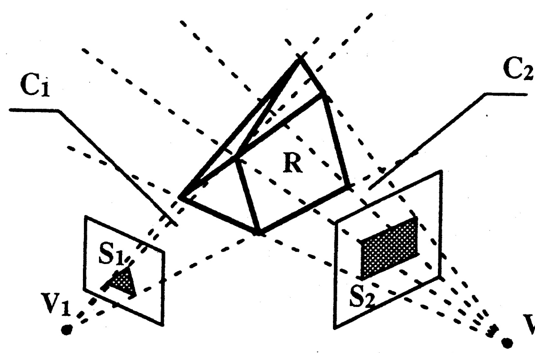

Fig.1–The volume intersection approach to the reconstruction of 3D objects.

perspective projection the regions C_i are cones obtained by back–projecting from a viewpoint V the corresponding silhouette. For orthographic projections, these regions are cylinders obtained by sweeping the silhouettes along lines parallel to the viewing directions. In both cases the regions are bounded by ruled surfaces. We refer to these surfaces as to the *circumscribed cones* or *cylinders* of **O**. In the following, when not otherwise explicitly stated, for simplicity we will speak of cones, conical surfaces and viewpoints referring both to perspective and parallel projections.

The rationale of the volume intersection approach to 3D object reconstruction is that silhouettes can usually be obtained with simple and robust algorithms from intensity images. Another feature of this approach is that it does not compel us to find correspondences between multiple images.

Many volume intersection algorithms specify the reconstructed object $\mathbf{R}_n$ with an octree representation(see for instance the work of Ahuja and Veenstra[7], Noborio et al.[13], Chien and Aggarwal[3]). Other representations have also been used, for instance by Kim and Aggarwal[2], and Martin and Aggarwal[10].

However, the volume intersection approach raises a number of theoretical questions, namely:
(a)which objects are exactly reconstructable;
(b)which is the closest approximation that can be obtained for non reconstructable objects;
(c)what can be inferred about the unknown object **O** from a reconstructed object **R**;
(d)how many silhouettes are necessary for the reconstruction;
(e)how to choose them efficiently.

The recently introduced geometric concept of visual hull answers questions (a) and (b)[18], [19], [20].

The inference of the shape of the unknown object from the reconstructed object(question (c)) is discussed in [21].

Choosing efficiently the silhouettes of the unknown object (question (e))is an important open problem. Some volume intersection algorithms use a fixed set of viewing directions(see for instance [4], [7], [14]). This provides certain advantages for intersection algorithms which specify the reconstructed object as an octree, but crude approximations of the unknown object may result. An object–specific, active approach could be more effective. Some work in this direction has been done by Shanmuck and Pujari[9] and by Lavakusha et al.[16]. To discuss the efficiency of the choice of the silhouettes requires a suitable definition of reconstruction accuracy. Such a definition, based on the visual hull concept, is provided by this paper.

As we will see in the following, only objects coincident with their visual hulls are exactly reconstructable, at least in theory. In practice, we must also consider the amunt of computation required for performing an exact reconstruction. This paper addresses the problem of finding the theoretical minimal number of intersections necessary for exactly reconstructing an object(question (d)). We are particularly interested in classes of objects which require a bounded number of intersections: even if optimal algorithms for choosing the silhouettes were available, we could exactly reconstruct only objects of these classes.

This paper is organized as follows. In Section II we describe the visual hull concept and summarize its relevant properties. In Section III we discuss a new definition of reconstruction accuracy. In Section IV we investigate the minimal number of intersections required for reconstructing an object (or its closest approximation).

II. The visual hull

The visual hull is a geometric entity which has been recently introduced by Laurentini[18],[19], [20], as a tool for dealing with silhouette–based image understanding, that is recognizing or reconstructing objects from their silhouettes. It provides immediate solutions to the first couple of problems stated in the introduction. In this section we summarize the material relevant to our discussion contained in [20].

The visual hull of a 3D object **O** can be informally described as the volume enveloped by all the possible circumscribed cones of **O**. A formal definition is as follows:

Definition 1

The visual hull ***VH(O,V)*** *of an object* ***O*** *relative to a viewing region* ***V*** *is a region of* E^3 *such that, for each point* $P \in$ ***VH(O,V)*** *and each viewpoint* $V \in$ ***V***, *the half line starting at V and passing through P contains at least a point of* ***O***.

Of course, it is **VH(O,V)≥O**. We also have that:

Proposition 1

If V>V', then VH(S,V)≤VH(S,V')

It can be shown that the following fundamental property holds:

Proposition 2

VH(O,V) *is the closest approximation of* ***O*** *that can be obtained using volume intersection techniques with viewpoints* $V \in$ ***V***.

This proposition is the answer to the second question stated in the introduction. From Proposition 2 we also obtain immediately the answer to the first question:

Proposition 3

An object ***O*** *can be exactly reconstructed by volume intersection techniques using silhouettes observed from a viewing region* ***V*** *if and only if it is* ***O=VH(O,V)***

An object satisfying the condition of Proposition 3 will be specified with the adjective *e–reco*.

Although this is not strictly relevant to our discussion, it is worth noting that **VH(O,V)** is also the largest object silhouette–equivalent(i.e., that gives the same silhouette) to **O** when observed from viewpoints belonging to **V**.

An object has an infinite number of visual hulls, one for each viewing region. However, we can restrict ourselves to two main cases.

The first case refers to any viewing region which completely encloses **O** without entering its convex hull. It can be shown that there is a unique visual hull for all these regions, which does not exceed the convex hull **CH(O)** of the object. This appears to be the case of main practical interest, since usually the object to be reconstructed lies at some distance and can assume any orientation with respect to the viewpoints. We refer to the visual hull relative to these regions as to the *external* visual hull, or simply the visual hull **VH(O)**, without any other specification. Examples of visual hulls of simple polyhedral objects are shown in Fig. 2.

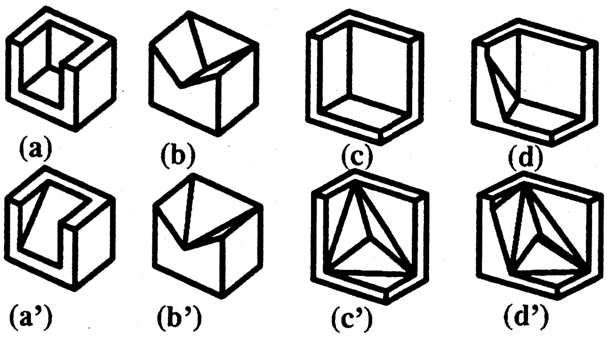

Fig.1–Four simple concave polyhedra(a), (b), (c), (d), and their visual hulls(a'), (b'), (c'), (d').

The second case, of lesser practical interest, refers to the unrestricted viewing region **V=E³–O**. The visual hull relative to this region is defined as the *internal* visual hull, and denoted **IVH(O)**.

Convex hull, visual hull, internal visual hull and O are related by the following inequalities:
Proposition 4

$O \leq IVH(O) \leq VH(O) \leq CH(O)$

Algorithms for computing **VH(O)** and **IVH(O)** have been given for polygonal sets[18], polyhedra[20] and solids of revolution[19].

III. Defining the reconstruction accuracy

It is clear that the choice of the number and the position of the viewpoints is a crucial point in volume intersection algorithms. For evaluating the effectiveness of this choice, it is necessary to define some accuracy index, that is some measure of the similarity between the original object **O** and the reconstructed object **R**. Since the volume intersection algorithm supplies volumes, a rather natural idea is to define the accuracy of reconstruction A_V as the ratio between the volume *Vol(***O***)* of the original object and the volume *Vol(***R***)* of the reconstructed object:

$A_V = Vol(\mathbf{O}) / Vol(\mathbf{R})$

This volumetric definition of accuracy has been assumed for instance by Ahuja and Veenstra [7], Potemsil[12], Noborio et al.[13], Shanmuk and Pujari[9], for evaluating the efficiency of their algorithms. However, this definition is not without shortcomings. One problem is that it mixes together two different kinds of reconstruction errors: those depending only on the intrinsic features of the object, and those related to the choice of the silhouette. The availability of the visual hull concept allows us to overcome this problem(which was perceived also by Ahuia and Veenstra[7]) by defining an accuracy A_{VH} which makes reference to the volume *Vol(***VH(O)***)* of the visual hull:

$A_{VH} = Vol(\mathbf{VH(O)}) / Vol(\mathbf{R})$

Let

$A_I = Vol(\mathbf{O}) / Vol(\mathbf{VH(O)})$

Then

$A_V = A_{VH} \times A_I$

where the accuracy A_I due to the intrinsic properties of **O** is separated from the accuracy A_{VH} related to the choice of the viewpoints.

However, let us remark that this definition is convenient for evaluating the reconstruction accuracy of *known* object. To construct effective *active* algorithms, at each step of the reconstruction process we need:

i) a measure of accuracy capable of indicating if a sufficiently precise reconstruction of an *unknown* object has already been obtained

ii) a rule for choosing a new silhouette which improves the accuracy when its current measure is unsatisfactory.

The volumetric accuracy measure A_{VH} is obviously unable to support this process, since it cannot be computed for the current object. To succeed in finding a new definition suitable for this purpose would be an important step toward effective active volume intersection algorithms.

IV. Minimal numbers of intersections necessary for exactly reconstructing some classes of objects or visual hulls

Let us suppose that, using the volume intersection technique, we attempt to achieve a unitary accuracy A_{VH}, that is to exactly reconstruct an object **O**, or, if **O** is not *e–reco*, its visual hull. Let IO_{min} in the former case, and IVH_{min} in the latter, be the theoretical minimal numbers of intersections that we must perform.

IO_{min} and IVH_{min} are important since they put computational limits to the possibility of exactly reconstructing in practice an object or its closest approximation. In this section we will discuss these numbers for some categories of objects. We are particularly interested in finding classes of objects with *bounded* IO_{min} or IVH_{min}. We will call *f–reco* the objects which in theory are exactly reconstructable with a finite number of intersections; if this property holds for their visual hulls, we shall call them *f–vhreco*. The viewpoints are supposed to lie outside the convex hull of the object. This assumption is required by the definition of visual hull, and in keeping with several practical situations of reconstruction for an *e–reco* object.

Let us recall that the volumes to be intersected have conical or cylindrical surfaces, produced by families of lines passing through a point or parallel to a direction. This means that the surface of any object obtained by means of a finite number of intersections consists of a finite number of patches of conical or cylindrical surfaces. Surfaces of this kind will be defined *R–surfaces*. We have

the following obvious statements:
i)–To have an *R–surface* is a necessary condition for an object to be *f–reco*.
ii)–To have a visual hull with an *R–surface* is a necessary condition for an object to be *f–vhreco*.

It is clear that the condition of statement *i)* is not a sufficient condition, since an object having an *R–surface* might not be coincident with its visual hull. The condition of statement *ii)* could be intuitively conjectured to also be sufficient: an example in this section will show that this is not the case.

From the above statements it follows that objects with curved surface are *f–reco* or *f–vhreco* only in exceptional cases. Even a simple curved object like a sphere is not *f–reco*. In these cases it seems preferable to speak of objects reconstructable with arbitrarily high accuracy *AVH*.

Polyhedral objects

Let us consider an apparently more promising category of objects, the polyhedra, which satisfy the above necessary condition for being *f–reco*, since their planar faces are *R–surfaces*. Let n be the number of faces of a polyhedron **P**; one could conjecture, at least for same classes of polyhedra, the existence of a function $f(n)$ such that IO_{min} or IVH_{min} are $O(f(n))$.

It is immediate to verify that this is true for the class of convex polyhedra, which obviously are *e–reco*. Actually, if the planes supporting the faces are in general position (a set of planes is said to be in general position if the intersection of any triplet of planes of the set is a point and all these points are different), we have that IO_{min} is the minimal integer which is larger or equal to $n/3$. This number, obtained by choosing the viewpoints at the intersections of disjoint triplets of planes supporting the faces, is an upper bound when the faces are not in a general position.

Let us consider now general polyhedra with n faces, which may or may not be *e–reco*. Let us consider first the class of objects different from their visual hull, and therefore not *e–reco*. We have presented examples of non–convex polyhedra (Fig.2) whose visual hulls are also polyhedra. However, there are concave polyhedra whose visual hulls are bounded by quadric ruled patches, generated by lines which are tangent to the polyhedron at three edges(see [18],[20]). An example of a such polyhedron is shown in Fig.3 , together with its visual hull. The curved patches are segments of hyperboloids of one sheet or hyperbolic paraboloids[20], which are not *R–surfaces*. We must therefore conclude that the class of general polyhedra is not *f–vhreco*. The same conclusion can be obtained by observing that, since every silhouette of a polyhedron is polygonal, and thus the volumes to intersect have planar faces, a curved patch cannot be generated with a finite number of intersections.

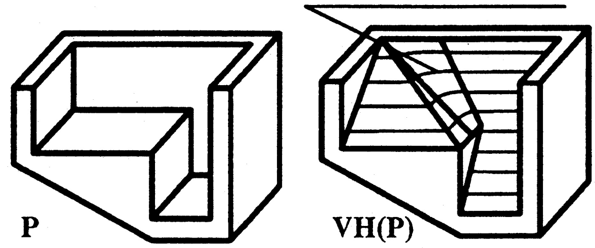

Fig.3–A polyhedron **P** and its visual hull **VH(P)**, whose surface contains a non–planar patch.

Let us now consider general polyhedra with a polyhedral visual hull. This class contains as a subclass the polyhedra *e–reco*. We have the rather surprising results that, using viewpoints lying outside the convex hull of the objects to reconstruct, this subclass is not *f–reco*, and the class of polyhedra with polyhedral visual hull is not *f–vhreco*. These counterintuitive statements can be proved by the following example. Let us consider the concave *e–reco* polyhedron(or visual hull)**PL** of Fig.4, with14

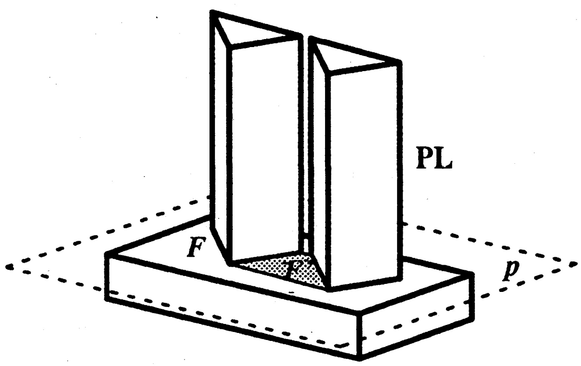

Fig.4– A polyhedron whose reconstruction may require an unbounded number of views.

faces. A section of **PL** made by the plane p supporting the face F is shown in Fig.5. To reconstruct F', the part of F highlighted in Fig.4, the viewpoints must lie on p in the regions R or R', which are outside the convex hull of the object. A possible viewpoint V and the part S of F' reconstructed from this viewpoint is shown in Fig.5. It is clear from the figure that, by reducing the distance between the vertical wedges without affecting the number of faces of the polyhedron, S can be made arbitrarily small, and therefore the number of silhouettes required for reconstructing F' arbitrarily large.

In conclusion, we have that:

Proposition 5
Using viewpoints outside the convex hull, an unbounded number of silhouettes may be required for exactly recon-

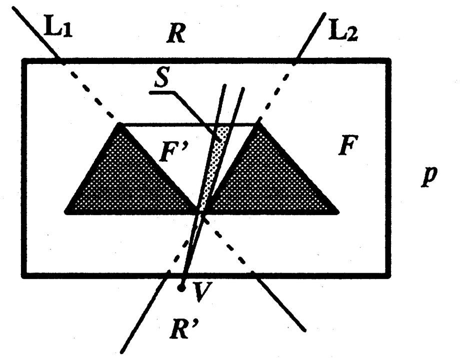

Fig.5– A section of the polyhedron **PL** made by the plane *p* supporting the face *F*.

structing a non–convex polyhedron or polyhedral visual hull with a bounded number of faces.

The example discussed also shows that to have a visual hull with an *R–surface* is not a sufficient condition for an object to be *f–vhreco*.

The viewing region is crucial for the number of intersections. In fact, let us now remove any restriction on the position allowed for the viewpoints, and consider the class of *e–reco* polyhedra . We have the important result that in this case the minimal number of intersections is bounded. More specifically, the following proposition holds:

Proposition 6

*Any e–reco polyhedron **PL** with n faces can be reconstructed with* $O(n^5)$ *volumetric intersections using unconstrained viewpoints.*

We will give a constructive proof of this statement, by describing an algorithm which for each face *F* of **PL** constructs $O(n^4)$ regions and $O(n^4)$ viewpoints such that:
i–each region can be reconstructed by a single viewpoint
ii–any point P of *F* belongs at least to one of these regions.

The algorithm for finding the viewpoints

The viewpoints necessary for exactly reconstructing a face *F* must lie in the plane *p* which supports *F*. In this theoretical study we will also admit viewpoints *lying* on *F*, even if this is a limit situation. Let us consider a point P of a face *F* supported by a plane *p*, and let **SP** be the polygonal intersection of **PL** and *p*, excluding *F* itself. If **SP** is empty, the face is convex and thus can be reconstructed with one intersection. Let us consider a not empty **SP**. Since by hypothesis each face is reconstructable, there must exist(at least) one line L passing through P and lying on *p* which shares with **PL** only points of *F*, and therefore does not cross at any point the boundary of **SP**. Let us rotate L clockwise about P. Line L will touch the boundary of **SP** at a point Q(see. Fig.6 (a)). Let us also rotate L counterclockwise until it touches the boundary of **SP** at a point R; there are two possible cases, shown in Fig.6.(b) and(c). Consider case (b), and rotate counterclockwise the line PQ about Q and clockwise the line PR about R, until they reach the boundary of **SP**(see Fig. 6(d), where the rotation of line PR is halted by one of the edges of **SP** converging at R). Choose as viewpoint the point V, lying at the intersection of the two rotated lines. From V, the zone highlighted in Fig. 6(d) can be reconstructed. In case (c), the viewpoint V' can be obtained by rotating counterclockwise both the lines PR and PQ, as shown in Fig. 6(e).

In conclusion, given a point P of *F* we have shown how to construct a viewpoint capable of reconstructing a

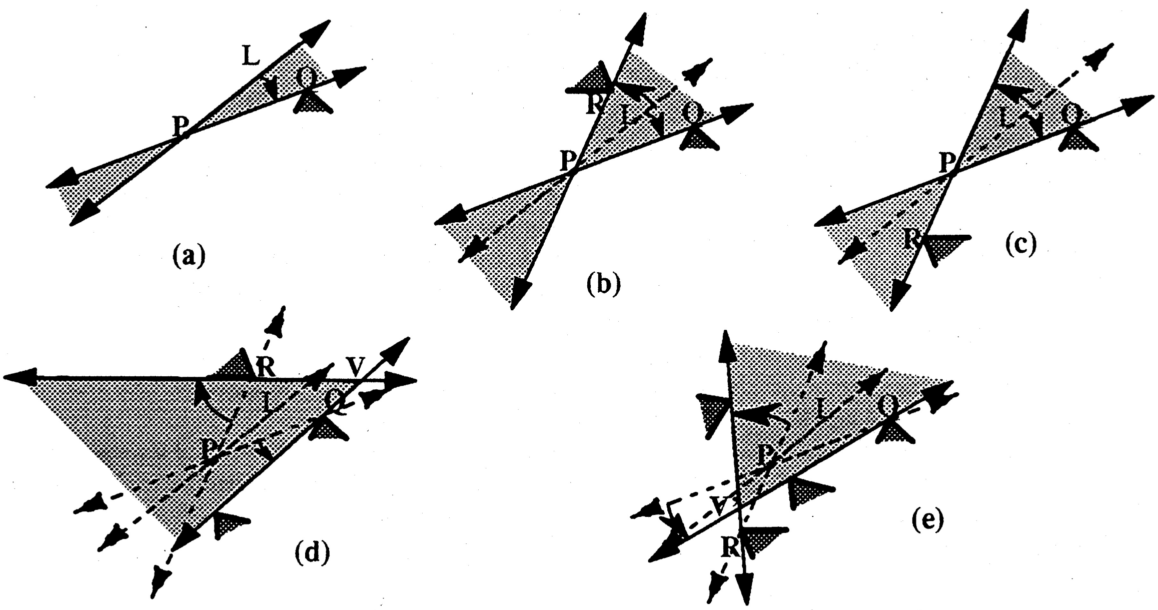

Fig.6–The construction of a viewpoint capable of reconstructing a zone of a face F containing a point P.

zone of F containing P. These viewpoints lie at the intersection of two lines passing trough two vertices of **SP** (this proposition has no exceptions if we also consider vertices belonging to the same edge of **SP**). These lines are $O(n^2)$, and thus all the possible viewpoints, which reconstruct at least once each point of the face, are $O(n^4)$. Considering all the faces of **PL**, we obtain the bound of $O(n^5)$ intersections.

Observe that the viewpoint constructed as shown before could lie everywhere on the plane of each face, and thus also *inside* the convex hull of **PL**. Let us consider again the example of Fig. 4. It is easy to see that the viewpoint suitable for reconstructing F' lies at the intersection of the lines L_1 and L_2, which is inside the convex hull of the object.

V. Summary

We have addressed some theoretical question raised by the volume intersection technique for reconstructing 3D objects. After introducing A_{VH}, a measure of reconstruction accuracy based on the concept of visual hull, we presented a discussion on the minimal number of volume intersection operations necessary for achieving unitary A_{VH}. This discussion is important, since it states computational limits to the exact reconstruction of an object using the volume intersection technique. Object with curved surfaces cannot be reconstructed with a bounded number of intersections, except for exceptional cases. Planar face objects are more likely to be exactly reconstructable with a bounded number of silhouettes. However, we have found that to exactly reconstruct a concave polyhedron, or its visual hull if the polyhedron is not reconstructable, using viewpoints lying outside the convex hull of the object may require an unbounded number of intersections. The number of intersections required is affected by the viewing region. Using unconstrained viewpoints, we have shown that any theoretically reconstructable polyhedron can actually be constructed with $O(n^5)$ intersections, and given an algorithm for finding the viewpoints .

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