Acyclic Hierarchical Cell Complexes

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Abstract

A hierarchical cell complex in $\mathbb{E}^n$ is a hierarchy of n-dimensional complexes such that each n-complex is the refinement of a cell belonging to its parent. In this paper, we define both hierarchical cell complexes and hierarchical simplicial complexes. Hierarchical simplicial complexes are used as a basis for multiresolution hypersurface modeling. We show that hierarchical complexes, in which each component is an acyclic complex, can be ordered back-to-front with respect to a viewpoint.

1 Introduction

A hierarchical cell complex $\mathcal{H}$ is a hierarchy of n-dimensional cell complexes in $\mathbb{E}^n$ such that each n-complex composing it is the refinement of a cell belonging to its parent complex in the hierarchy. Thus, a hierarchical complex can be viewed as a description of a cell complex at different degrees of abstraction. A special class of hierarchical cell complexes is provided by hierarchical simplicial complexes, which are basically hierarchies of simplicial complexes in $\mathbb{E}^n$. The motivation which has led us to the definition of hierarchies of complexes is the development of multiresolution representations of hypersurfaces describing multidimensional scalar fields for specific applications in volume visualization. When the data are available at irregularly spaced points, simplicial meshes connecting the data points are used as basic connection structure [Mes91,Wil92].

In volume visualization or spatial data analysis, large quantities of data are generally available. Thus, it is desirable to have a multiresolution model of a hypersurface which acts as a data compression mechanism on the data. To this aim, we have defined a hierarchical model of a hypersurface (i.e., a surface described by a function $z = f(x_1, \ldots, x_n)$) as a hierarchy of hypersurface models based on simplicial complexes [Ber93]. This model extends a hierarchical representation for surfaces defined in [DeF92] as a digital elevation model. Hierarchical surface models have been developed in the literature for describing terrains at different levels of resolution (see [DeF92]). No such models have been defined for three or more dimensions.

For visualization purposes (in particular direct volume rendering based on projective methods), it is fundamental that the cells of the complex used as connecting mesh for the data points can be ordered with respect to any given viewpoint. A visibility order of a set of objects from some viewpoint is a sequence such that, if object A obstructs object B, then B preceeds A in the sequence. Such order exists if the obstruction relation, usually called *infront/behind* relation, is acyclic with respect to the viewpoint. Edelsbrunner proved [Ede90] that the *infront/behind* relation is acyclic for all cell complexes in $\mathbb{E}^n$ that can be obtained by projecting the boundary complex of a convex polytope in $\mathbb{E}^{n+1}$. The Delaunay triangulation of any finite point set, for instance, belongs to such class of complexes.

In the paper, we prove that the *infront/behind* relation on any cell complex, obtained by partially or fully expanding a hierarchical cell complex, is acyclic, provided that each complex forming the hierarchical cell complex be acyclic. In particular, the expansion of Delaunay-based hierarchical simplicial complexes produces acyclic cell complexes.

2 Preliminaries

This Section introduces some definitions and notations.

Informally, a cell complex is a subdivision of an n-

dimensional topological space into cells. Here, we consider cell complexes in the n-dimensional Euclidean space $\mathbb{E}^n$.

A *k-cell* in $\mathbb{E}^n$, with $0 \leq k \leq n$, is a closed connected subset of $\mathbb{E}^n$, such that its interior is homeomorfic to an open k-dimensional disk and whose boundary is not null. We denote with $b(\gamma)$ and $i(\gamma)$ the boundary and the interior, respectively, of a k-cell γ. A k-cell γ is said to be a cell of *order k*. A q-cell γ' is said to be a *q-face* (or, simply, a *face*) of a cell γ if and only if $\gamma' \subseteq \gamma$. A face γ' of a cell γ is called a *proper face* of γ if and only if $\gamma' \subset \gamma$. The set $\{\gamma' \mid \gamma' \ s\text{-}face\ of\ \gamma,\ s \leq k\}$ is called the *s-skeleton* of γ, and denoted with $\gamma^{(s)}$.

A **cell complex** Γ of order n in $\mathbb{E}^n$ (also called an **n-complex**) is a finite collection of cells satisfying the following properties:

(i) for each pair $\gamma_1, \gamma_2 \in \Gamma$, where γ_1 is a p-cell, with $p \leq n$, and γ_2 an m-cell, with $m \leq n$, $i(\gamma_1) \cap i(\gamma_2) = \emptyset$;

(ii) the boundary $b(\gamma)$ of a cell $\gamma \in \Gamma$ is the union of cells belonging to Γ;

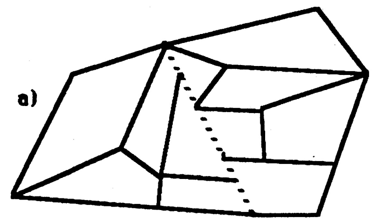
(iii) for each pair $\gamma_1, \gamma_2 \in \Gamma$ such that $\gamma_1 \cap \gamma_2 \neq \emptyset$, $\gamma_1 \cap \gamma_2$ is the union of cells of Γ.

Let Γ be an n-complex. The set $\{\gamma' \mid \gamma' \ s\text{-}cell\ of\ \Gamma\}$, with $0 \leq s \leq n$, is called the *s-skeleton* of Γ and denoted with $\Gamma^{(s)}$. Thus, an n-complex Γ is described by the $(n+1)$-tuple $\Gamma = (\Gamma^{(0)}, \ldots, \Gamma^{(n)})$ of its s-skeletons, with $s = 0, \ldots, n$. The union of all the s-cells of Γ, $0 \leq s \leq n$, regarded as point sets, is called the *domain* of Γ and denoted $D(\Gamma)$.

Let Γ_i and Γ_j be two n-complexes and let $\gamma_1 \in \Gamma_i^{(n)}$ and $\gamma_2 \in \Gamma_j^{(n)}$. Cells γ_1 and γ_2 are said to be *s-adjacent*, with $1 \leq s \leq n-1$, if and only if there exists an s-cell γ_p such that $\gamma_1 \cap \gamma_2 \equiv \gamma_p$ and γ_p is union of proper faces of γ_1 or of proper faces of γ_2 (in this case, γ_1 and γ_2 are said to be *adjacent along γ_p*). If V_1 and V_2 are the sets of vertices of γ_1 and γ_2, respectively, then γ_1 and γ_2 are said to be *matching* along an s-cell γ_p if and only if γ_1 and γ_2 are s-adjacent along γ_p and $V_1 \cap \gamma_p \equiv V_2 \cap \gamma_p$, where γ_p is the union of proper faces of both γ_1 and γ_2.

Let Γ_i and Γ_j be two n-complexes and $\gamma \equiv D(\Gamma_i) \cap D(\Gamma_j)$ be an s-cell, with $1 \leq s \leq n-1$. Γ_i and Γ_j are said to be *s-adjacent* along γ if and only if γ is union of s-cells, each of which is the intersection of an s-cell of Γ_i and of an s-cell of Γ_j. Moreover, if Γ_i and Γ_j are s-adjacent along γ, then Γ_i and Γ_j are said to be *matching* if and only if $\gamma \cap V_i \subseteq \gamma \cap V_j$ or $\gamma \cap V_j \subseteq \gamma \cap V_i$ and, for each pair γ_1, γ_2 with $\gamma_1 \in \Gamma_i^{(n)}$ and $\gamma_2 \in \Gamma_j^{(n)}$, such that $\gamma_1 \cap \gamma_2 \equiv \gamma'$ ($\gamma' \subseteq \gamma$), γ' is union of proper faces of γ_1 or of proper faces of γ_2. This latter condition guarantees that the intersection of two n-cells belonging to different matching complexes is always union of cells which are all proper faces of one of the two n-cells. Figure 1 shows an example of 1-adjacent 2-complexes and of matching 2-complexes.

a)



b)

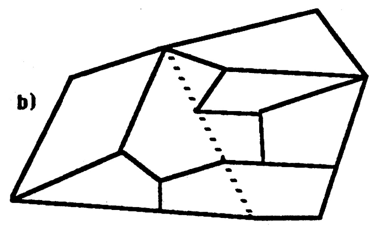


Figure 1: a) An example of 1-adjacent 2-complexes. b) An example of matching 2-complexes.

An n-complex Γ is called *convex* if and only if $D(\Gamma)$ is a convex set and also the cells of Γ are convex sets. Two cells γ_1 and γ_2 of an n-complex Γ are said to be *s-adjacent*, with $s < n$, if and only if there exists an s-cell γ which is a proper face of both γ_1 and γ_2. Note that γ_1 and γ_2 are matching along s-cell γ. Two cells γ_1 and γ_2 are said to be *s-connected* if and only if there exists a sequence $[\gamma'_1, \ldots, \gamma'_k]$ of s-cells of Γ such that $\gamma'_1 \equiv \gamma_1$, $\gamma'_k \equiv \gamma_2$ and, $\forall\, i = 1, \ldots, k-1$, γ'_i and γ'_{i+1} are s-adjacent. An n-complex Γ is said to be *s-connected*, with $1 \leq s \leq n-1$, if and only if, for each partition of Γ in two subsets Γ_i and Γ_j, there exist $\gamma_1 \in \Gamma_i^{(n)}$ and $\gamma_2 \in \Gamma_j^{(n)}$ such that γ_1 and γ_2 are s-connected. In the following, we will always consider convex, $(n-1)$-connected n-complexes, with polyhedral cells.

All definitions given above for cell complexes can be specialized for simplicial complexes. A **simplicial complex** of order n (also called an **n-simplicial complex**) in $\mathbb{E}^n$ is a collection Σ of simplices satisfying the following properties:

(i) for each pair $\sigma_1, \sigma_2 \in \Sigma$, where σ_1 is a p-simplex, with $p \leq n$, and σ_2 an m-simplex, with $m \leq n$, $i(\sigma_1) \cap i(\sigma_2) = \emptyset$;

(ii) the boundary $b(\sigma)$ of a simplex $\sigma \in \Sigma$ is the union of simplices belonging to Σ;

(iii) for each pair $\sigma_1, \sigma_2 \in \Sigma$ such that $\sigma_1 \cap \sigma_2 \neq \emptyset$, $\sigma_1 \cap \sigma_2$ is a simplex of Σ.

The class of simplicial complexes can be enlarged by extending the definition of a k-simplex to that of a generalized k-simplex. A *generalized k-simplex* can be either a k-simplex, if it has exactly $k+1$ vertices, or a k-cell, with $m > k+1$ vertices, in which exactly $k+1$ vertices form a k-simplex σ and the other $m-k-1$ vertices lie on the boundary of σ. A **generalized simplicial complex** Σ of order n in $\mathbb{E}^n$ is a cell complex of order n such that every cell is a generalized simplex.

Let Γ be an n-complex, and $\gamma_1, \gamma_2 \in \Gamma^{(n)}$ such that $\gamma_1 \neq \gamma_2$. γ_2 **obstructs** γ_1 with respect to a given viewpoint v_p in $\mathbb{E}^n$ if and only if there exists a ray r emanating from v_p such that $r \cap i(\gamma_1) \neq \emptyset$ and $r \cap i(\gamma_2) \neq \emptyset$ and, $\forall\, p \in r \cap i(\gamma_2)$ and $\forall\, p' \in r \cap i(\gamma_1)$, p lies between v_p and p'.

Let Γ be an n-complex, and $\gamma_1, \gamma_2 \in \Gamma^{(n)}$ such that $\gamma_1 \neq \gamma_2$. If γ_1 and γ_2 are $(n-1)$-adjacent, an order between the two cells with respect to point v_p is defined as follows: $\gamma_1 <_{v_p} \gamma_2$ if and only if γ_2 obstructs γ_1. Such relation is called **infront/behind** relation. The transitive closure of $<_{v_p}$ is denoted with $<^*_{v_p}$.

A *cycle* in Γ with respect to v_p is a sequence of n-cells $[\gamma_1, \gamma_2, \ldots, \gamma_k]$ such that $\gamma_1 <_{v_p} \gamma_2 <_{v_p}, \ldots, <_{v_p} \gamma_k <_{v_p} \gamma_1$. The *infront/behind* relation on Γ is *acyclic* if and only if, for every viewpoint, no cycle exists. In this case, Γ is said to be an *acyclic complex*. A *visibility order* of an acyclic n-complex Γ with respect to a viewpoint v_p is a linear sequence of the n-cells of Γ such that if the cell γ_2 obstructs the cell γ_1, then γ_1 preceeds γ_2 in the sequence. Any total order, that is consistent with the order defined by the *infront/behind* relation on Γ, is a visibility order of the cells of Γ relative to v_p (i.e., for any pair of cells γ_1 and γ_2, if γ_2 obstructs γ_1, then $\gamma_1 <^*_{v_p} \gamma_2$).

If a viewpoint v_p does not lie on any hyperplane containing an $(n-1)$-face of a complex Γ, then it is possible to relate any pair of n-cells through relation $<_{v_p}$. If v_p lies on such hyperplane, then it can be infinitesimally perturbed in order to generate rays which intersect the interiors of all the cells of Γ. An alternative way to solve such degeneracies is to establish an arbitrary order between the pairs of n-cells for which the *infront/behind* relation cannot be stated.

3 Hierarchical Cell Complexes

In this Section, we introduce hierarchical cell complexes as hierarchies of cell complexes in $\mathbb{E}^n$, and the concepts of fully and partially expanded cell complex associated with such complexes.

Let $\mathcal{N} = \{\Gamma_0, \ldots, \Gamma_h\}$ be a family of n-complexes such that, $\forall\, i = 0, \ldots, h$, $\exists\, \gamma_i \in \cup_{j=0, j\neq i}^{h} \Gamma_j^{(n)}$, with $\gamma_i \equiv D(\Gamma_i)$. A **hierarchy of cell complexes** of order n is a family $\mathcal{N}$ of n-complexes described by a tree $\mathcal{H} = (\mathcal{N}, \mathcal{E})$, where $\mathcal{E} = \{(\Gamma_i, \Gamma_j) \mid \Gamma_i, \Gamma_j \in \mathcal{N}$ *and* $\exists\,!\, \gamma_j \in \Gamma_i^{(n)}$ *such that* $\gamma_j \equiv D(\Gamma_j)$ *and* $\Gamma_j^{(n)} \neq \{\gamma_j\}\}$. Each arc $(\Gamma_i, \Gamma_j) \in \mathcal{E}$ is labeled with γ_j.

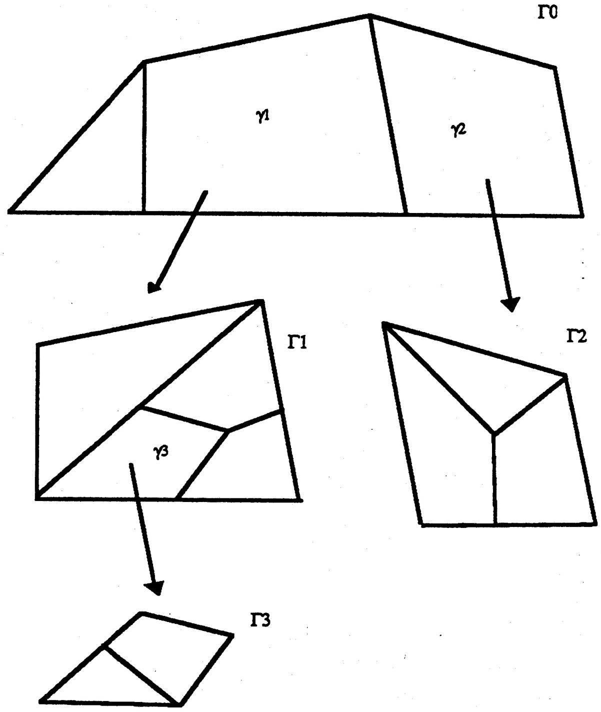


Figure 2: A hierarchical cell complex of order 2.

Let $\mathcal{H} = (\mathcal{N}, \mathcal{E})$ be a hierarchy of cell complexes of order n. Any $\Gamma_i \in \mathcal{N}$ is called a *component* of $\mathcal{H}$. Let Γ_i and Γ_j be two components of $\mathcal{H}$. Γ_j is said to be a *child* of Γ_i and Γ_i is said to be the *parent* of Γ_j if and only if $(\Gamma_i, \Gamma_j) \in \mathcal{E}$. Similarly, the *descendants* of Γ_i are all those complexes Γ_k which are linked to Γ_i by a path in $\mathcal{H}$ from Γ_i to Γ_k, while the *ancestors* of Γ_j are all the complexes Γ_p which are linked to Γ_j by a path from Γ_p to Γ_j. If $\gamma_j \in \Gamma_i^{(n)}$ and $\Gamma_j \neq \Gamma_0$, then Γ_j is said to be the *direct expansion* of γ_j (and γ_j the *direct abstraction* of Γ_j) if and only if $D(\Gamma_j) \equiv \gamma_j$ and Γ_j is a child of Γ_i. In this case, cell γ_j is called a *macrocell*. All complexes Γ_q linked to Γ_i by a path $[\Gamma_i, \Gamma_{k_1}, \ldots, \Gamma_{k_m}, \Gamma_q]$ in $\mathcal{H}$, where Γ_{k_1} is the direct expansion of a macrocell $\gamma_j \in \Gamma^{(n)}$, are said to be *expansions* of γ_j, while all n-cells of such complexes are said to be *descendant cells* of γ_j.

A **Hierarchical Cell Complex (HCC)** of order n is a hierarchy of cell complexes $\mathcal{H} = (\mathcal{N}, \mathcal{E})$ of order n, in which any two complexes Γ_i and Γ_j of $\mathcal{H}$ adjacent along an s-cell γ $(1 \leq s < n)$ are matching along γ. Figure 2 shows an example of a hierarchical cell complex of order 2.

Let $\mathcal{H} = (\mathcal{N}, \mathcal{E})$ be an HCC. Let $\mathcal{N}'$ be a proper subset of

$\mathcal{N}$ and $\mathcal{E}'$ be the set of arcs of $\mathcal{E}$ having their endnodes in $\mathcal{N}'$. If $\mathcal{H}' = (\mathcal{N}', \mathcal{E}')$ is a tree, then $\mathcal{H}'$ is called a *hierarchical subcomplex* of $\mathcal{H}$.

An HCC $\mathcal{H}$ defines a hierarchical representation of a cell complex $\Gamma_{\mathcal{H}}$, called the *expanded complex* associated with $\mathcal{H}$. Such complex is defined on the basis of the process of cell refinement. In more details, the process of replacing a macrocell γ_j in an n-complex Γ_i with an n-complex Γ_j such that $D(\Gamma_j) \equiv \gamma_j$ is called *refinement*. The complex Γ_{ij} obtained by refinement of γ_j with Γ_j in Γ_i is defined as follows:

$$\Gamma_{ij}^{(s)} = \Gamma_i^{(s)} \setminus \gamma_j^{(s)} \cup \Gamma_j^{(s)}, \ (0 \le s \le n),$$

where $\gamma_j^{(s)}$ is the s-skeleton of γ_j and $\Gamma_{ij}^{(s)}$, $\Gamma_i^{(s)}$ and $\Gamma_j^{(s)}$ are the s-skeletons of Γ_{ij}, Γ_i and Γ_j, respectively. We can, thus, define a function ρ such that $\rho(\Gamma_i, \Gamma_j, \gamma_j) = \Gamma_{ij}$.

Thus, the expanded complex $\Gamma_{\mathcal{H}}$ associated with $\mathcal{H}$ is the cell complex obtained by iteratively refining all the macrocells belonging to the complexes in the hierarchy. More formally, we introduce the function ρ_T, such that $\rho_T(\mathcal{H}) = \Gamma_{\mathcal{H}}$, defined as follows:

$$\rho_T(\mathcal{H}) = \begin{cases} \Gamma_0 \ \ \underline{if} \ \ \mathcal{H} = (\{\Gamma_0\}, \emptyset) \\ \\ \rho(\ldots(\rho(\Gamma_0, \rho_T(\mathcal{H}_{k_1}), \gamma_1)\ldots, \rho_T(\mathcal{H}_{k_p}), \gamma_p), \\ \quad with\ \gamma_r \equiv D(\Gamma_{k_r}),\ r = 1, \ldots, p \\ \quad \underline{otherwise} \end{cases}$$

where Γ_0 is the root of $\mathcal{H}$, $\Gamma_{k_1}, \ldots, \Gamma_{k_p}$ are all the children of Γ_0 and $\mathcal{H}_{k_i}$, $i = 1, \ldots, p$, is the subtree of $\mathcal{H}$ rooted at Γ_{k_i}. It has been shown that both function ρ and function ρ_T define a cell complex [Ber93]. Thus, the cell complex $\Gamma_{\mathcal{H}}$ defined by function ρ_T applied to an HCC $\mathcal{H}$ is said to be the *(fully) expanded complex associated with* $\mathcal{H}$.

As we have already seen, an HCC $\mathcal{H}$, rooted at a complex Γ_0, provides a hierarchical description of the refinements of the cells of Γ_0. Sometimes, only some of these refinements are of interest. Thus, we call *partially expanded complex* associated with $\mathcal{H}$ any (fully) expanded complex associated with a hierarchical subcomplex of $\mathcal{H}$. On the basis of this definition, if $\mathcal{H}'$ is a hierarchical subcomplex of $\mathcal{H}$, then the partially expanded complex associated with $\mathcal{H}$, obtained by full expansion of $\mathcal{H}'$, is denoted with $\Gamma_{\mathcal{H}'}$.

4 Hierarchical Simplicial Complexes

In this Section, we consider hierarchies of complexes whose components are simplicial complexes. We define two different types of hierarchical complexes, that we term *generalized hierarchical simplicial complexes* and *hierarchical simplicial complexes*, based on two different definitions of matching, called *weak* and *strong* matching, respectively.

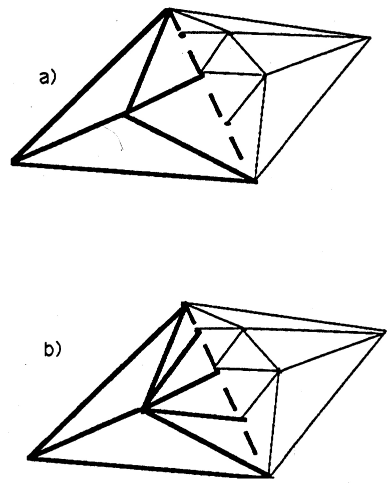


Figure 3: a) Weakly matching simplicial complexes of order 2. b) Strongly matching simplicial complexes of order 2.

Let $\Sigma_i = (\Sigma_i^{(0)}, \ldots, \Sigma_i^{(n)})$ and $\Sigma_j = (\Sigma_j^{(0)}, \ldots, \Sigma_j^{(n)})$ be two n-simplicial complexes and V_i and V_j their 0-skeletons (i.e., $V_i = \Sigma_i^{(0)}$ and $V_j = \Sigma_j^{(0)}$). If Σ_i and Σ_j are s-adjacent along an s-simplex σ, then Σ_i and Σ_j are said to be *weakly matching* along σ if and only if

- $\sigma \cap V_i \subseteq \sigma \cap V_j$ or $\sigma \cap V_j \subseteq \sigma \cap V_i$;
- For each pair σ_i, σ_j with $\sigma_i \in \Sigma_i^{(n)}$ and $\sigma_j \in \Sigma_j^{(n)}$, such that $\sigma_i \cap \sigma_j \equiv \sigma'$ ($\sigma' \subseteq \sigma$), σ' is a proper face of σ_i or σ_j.

Two n-simplicial complexes Σ_i and Σ_j are *coincident*, i.e., $\Sigma_i \equiv \Sigma_j$, if and only if they are composed of the same simplices. Two n-simplicial complexes Σ_i and Σ_j, which are s-adjacent along an s-simplex σ, are said to be *strongly matching* along σ if and only if

- $\sigma \cap V_i \equiv \sigma \cap V_j$;
- Let $V_{ij} = \sigma \cap V_i \equiv \sigma \cap V_j$ and let Σ_r', $r = i, j$, be the s-complex whose s-skeleton is V_{ij} such that $D(\Sigma_r') \equiv \sigma$ and composed of simplices of Σ_r. Then $\Sigma_i' \equiv \Sigma_j'$.

Figure 3 shows examples of weakly and strongly matching simplicial complexes. It has been proven [Ber93] that, if Σ_i and Σ_j are two strongly matching simplicial complexes, then they are weakly matching as well.

Let $\mathcal{S} = \{\Sigma_0, \dots, \Sigma_h\}$ be a family of n-simplicial complexes such that $\forall\ i = 1, \dots, h,\ \exists\ \sigma_i \in \cup_{j=0, j\neq i}^{h} \Sigma_j^{(n)}$, with $\sigma_i \equiv D(\Sigma_i)$. **A hierarchy of simplicial complexes** of order n is a family $\mathcal{S}$ described by a tree $\mathcal{H} = (\mathcal{S}, \mathcal{E})$, where $\mathcal{E} = \{(\Sigma_i, \Sigma_j) \mid \Sigma_i, \Sigma_j \in \mathcal{S}\ and\ \exists\,!\ \sigma_j \in \Sigma_i^{(n)}\ such\ that\ \sigma_j \equiv D(\Sigma_j)\ and\ \Sigma_j^{(n)} \neq \{\sigma_j\}\}$. Each arc $(\Sigma_i, \Sigma_j) \in \mathcal{E}$ is labeled with σ_j.

A Generalized Hierarchical Simplicial Complex (GHSC) of order n is a hierarchy of simplicial complexes $\mathcal{H} = (\mathcal{S}, \mathcal{E})$ of order n in which any two simplicial complexes Σ_i and Σ_j of $\mathcal{H}$ adjacent along an s-simplex σ $(1 \leq s < n)$ are weakly matching along σ.

A Hierarchical Simplicial Complex (HSC) of order n is a hierarchy of simplicial complexes $\mathcal{H} = (\mathcal{S}, \mathcal{E})$ of order n satisfying the two following conditions:

(i) for every pair Σ_i, Σ_j of complexes of $\mathcal{S}$, adjacent along an s-simplex σ $(1 \leq s \leq n-1)$, with V_i and V_j 0-skeletons of Σ_i and Σ_j, respectively, one of the following conditions must hold:

- Σ_i and Σ_j are strongly matching along σ;
- Σ_i and Σ_j are weakly matching along σ, with $V_i \cap \sigma \subset V_j \cap \sigma$ and there exists a descendant of Σ_i which is strongly matching with Σ_j along σ;
- Σ_i and Σ_j are weakly matching along σ, with $V_j \cap \sigma \subset V_i \cap \sigma$ and there exists a descendant of Σ_j which is strongly matching with Σ_i along σ;

(ii) for every unexpanded n-simplex σ_i and for every macrosimplex σ_j adjacent to σ_i along an s-simplex σ_p, for all descendant simplices σ_r of σ_j such that $\sigma_r \cap \sigma_p \neq \emptyset$ and $\sigma_r \cap \sigma_p$ is not on the boundary of σ_p, then $\sigma_r \cap \sigma_p \equiv \sigma_p$.

Condition *(ii)* simply means that all macrosimplices, which are $(n-1)$-adjacent to an unexpanded n-simplex σ, must not modify the face they have in common with σ. The strongly matching rule, which characterizes an HSC, ensures that each macrosimplex of the subtree rooted at a simplicial complex Σ_k is expanded into a complex whose boundary faces are shared by its adjacent complexes.

Note that the class of the HSCs is a proper subclass of that of the GHSCs. It has been proven that the expanded complex associated with a GHSC is a generalized simplicial complex, while the one associated with an HSC is always a simplicial complex [Ber93]. On the other hand, partially expanded complexes associated with an HSC can either be simplicial complexes or generalized ones.

Because of its importance as a basis for a surface model, we have considered a special class of GHSCs and HSCs, i.e., those based on Delaunay simplices. A Delaunay n-complex Σ of a set of points $V \in \mathbb{E}^n$ is a convex, $(n-1)$-connected simplicial complex such that the circumsphere of any simplex $\sigma \in \Sigma$ does not contain any vertex of V inside. A (generalized) hierarchical simplicial complex, in which each component is a Delaunay simplicial complex, is called a *Delaunay* HSC (GHSC).

5 Acyclicity of Hierarchical Cell Complexes

In this Section, we show that any (partially or fully) expanded complex associated with a hierarchical cell complex, in which each complex is acyclic as well, is acyclic, by extending the result by Edelsbrunner on the acyclicity of n-dimensional projective cell complexes [Ede90].

An n-dimensional HCC $\mathcal{H}$ is said to be *acyclic* if and only if each of its component is acyclic. Theorem 5.3 relates the acyclicity of $\mathcal{H}$ with that of the expanded complex associated with it. To prove Theorem 5.3, we need a result which ensures the acyclicity of a complex generated through the refinement process. Lemma 5.1 shows that, if an n-cell γ_2 of an n-complex Γ_i obstructs one of the n-cells belonging to the direct expansion of an n-cell γ_1 of Γ_i $(n-1)$-adjacent to γ_2, then γ_2 obstructs γ_1.

Lemma 5.1 *Let v_p be a viewpoint. Let Γ_i be an acyclic cell complex of order n. Let $\gamma_1, \gamma_2 \in \Gamma_i^{(n)}$ such that γ_1 and γ_2 are $(n-1)$-adjacent. Let Γ_j be an n-complex such that $\gamma_1 \equiv D(\Gamma_j)$. Let Γ_{ij} be the cell complex obtained by refining γ_1 in Γ_i with Γ_j. If there exists an n-cell $\gamma'_k \in \Gamma_j^{(n)}$ such that $\gamma'_k <_{v_p} \gamma_2$, then $\gamma_1 <_{v_p} \gamma_2$.*

Proof. Since $\gamma'_k <_{v_p} \gamma_2$, there exists a ray r_k emanating from v_p such that $r_k \cap i(\gamma'_k) \neq \emptyset$ and $r_k \cap i(\gamma_2) \neq \emptyset$ and $\forall\ p_k \in r_k \cap i(\gamma_2)$, p_k lies between v_p and p'_k, $\forall\ p'_k \in r_k \cap i(\gamma'_k)$. Let $p_2 \in r_k \cap i(\gamma_2)$ and $p'_k \in r_k \cap i(\gamma'_k)$; p_2 lies between v_p and p'_k. Let us assume that $\gamma_2 <_{v_p} \gamma_1$. As Γ_i is acyclic, $\forall\ r$, such that $r \cap i(\gamma_1) \neq \emptyset$ and $r \cap i(\gamma_2) \neq \emptyset$ we have that $\forall\ p \in r \cap i(\gamma_1)$, p lies between v_p and p', $\forall\ p' \in r \cap i(\gamma_2)$. But there exists ray r_k such that $r_k \cap i(\gamma_1) \neq \emptyset$ (since $r_k \cap i(\gamma'_k) \subseteq r_k \cap i(\gamma_1)$) and $r_k \cap i(\gamma_2) \neq \emptyset$ and, moreover, $p'_k \in r_k \cap i(\gamma_1)$, does not lie between v_p and $p_2 \in r_k \cap i(\gamma_2)$. □

Similarly, we can prove that if an n-cell γ_2 of an n-complex Γ_i is obstructed by one of the n-cells belonging to the direct expansion of an n-cell γ_1 of Γ_i $(n-1)$-adjacent to γ_2, then γ_1 obstructs γ_2.

Lemma 5.2 shows that the refinement of an n-cell in an acyclic n-complex with an acyclic n-complex preserves the *infront/behind* relation.

Lemma 5.2 *Let Γ_i and Γ_j be two cell complexes of order n. Let $\gamma_j \in \Gamma_i^{(n)}$. Let Γ_{ij} be the cell complex obtained by refining γ_j in Γ_i with Γ_j. If Γ_i and Γ_j are acyclic, then Γ_{ij} is acyclic.*

Proof. Let us assume that there exists a simple cycle $\overline{S} = [\overline{\gamma}_1, \ldots, \overline{\gamma}_z]$ in Γ_{ij}, i.e., for some viewpoint v_p, $\overline{\gamma}_1 <_{v_p} \overline{\gamma}_2 <_{v_p} \cdots <_{v_p} \overline{\gamma}_z <_{v_p} \overline{\gamma}_1$. Cells $\overline{\gamma}_1, \ldots, \overline{\gamma}_z$ do not all belong to Γ_i or to Γ_j (otherwise either Γ_i or Γ_j would not be acyclic). Let us now consider the ordered sequence S obtained from $\overline{S}$ by substituting all the cells of Γ_j with γ_j. S defines a cycle of cells, all belonging to Γ_i, because of Lemma 5.1. In fact, let $\overline{\gamma}_k, \overline{\gamma}_{k+1} \in \overline{S}$ such that $\overline{\gamma}_k \in \Gamma_i$ and $\overline{\gamma}_{k+1} \in \Gamma_j$; if $\overline{\gamma}_k <_{v_p} \overline{\gamma}_{k+1}$, then $\overline{\gamma}_k <_{v_p} \gamma_j$. This contradicts the hypothesis that Γ_i is acyclic. Hence, Γ_{ij} must be acyclic. □

Theorem 5.3 *Let $\mathcal{H} = (\mathcal{N}, \mathcal{E})$ be a hierarchical cell complex of order n such that $\mathcal{N} = \{\Gamma_0, \Gamma_1, \ldots, \Gamma_h\}$, rooted at Γ_0. If $\mathcal{H}$ is acyclic, then the fully expanded cell complex $\Gamma_{\mathcal{H}}$ associated with $\mathcal{H}$ is acyclic.*

Proof. We show that if, for every $i = 0, \ldots, h$, Γ_i is acyclic, then $\Gamma_{\mathcal{H}}$ is acyclic. Let $\mathcal{H}'$ be a hierarchical subcomplex of $\mathcal{H}$ rooted at Γ_0. The proof is done by induction on the number of nodes of $\mathcal{H}'$. Let $\Gamma_{\mathcal{H}'}$ be the expanded complex associated with $\mathcal{H}'$. We must prove that $\Gamma_{\mathcal{H}'}$ is acyclic. If $\mathcal{H}'$ has only one node, then $\Gamma_{\mathcal{H}'}$ is the same as Γ_0, which is acyclic. We assume that every hierarchical subcomplex of $\mathcal{H}$ rooted at Γ_0 with less than s nodes has an expanded complex associated which is acyclic. Let $\mathcal{H}''$ be one of such subcomplexes and $\Gamma_{\mathcal{H}''}$ the expanded complex associated with it, and let $\mathcal{H}'$ be the hierarchical subcomplex of $\mathcal{H}$ with exactly s nodes such that $\Gamma_{\mathcal{H}'}$ corresponds to the refinement of a macrocell in $\Gamma_{\mathcal{H}''}$. To prove that we obtain an acyclic complex, it is enough to show that the process of refining a macrocell γ_j in an acyclic complex with another acyclic complex preserves acyclicity. This is a consequence of Lemma 5.2. □

The following result is an immediate consequence of Theorem 5.3.

Corollary 5.4 *Let $\mathcal{H} = (\mathcal{N}, \mathcal{E})$ be an acyclic hierarchical cell complex of order n. Any partially expanded complex associated with $\mathcal{H}$ is acyclic.*

A visibility order of the cells in a cell complex resulting from the expansion of a hierarchical cell complex can be obtained by combining a tree traversal with a topological sorting algorithm applied to each complex, as shown in [Ber93].

6 Concluding Remarks

We have presented a new hierarchical representation for cell complexes in n dimensions, called a hierarchical cell complex. We have also defined hierarchical simplicial complexes for a multiresolution representation of a hypersurface in $\mathbb{E}^{n+1}$. In particular, we are interested in the application of such representation to volume data description at different levels of resolution. An important requirement in such application is the capability of sorting the cells forming the complex, used as basic domain discretization, in a back-to-front order. To this aim, we have proven the acyclicity of the *infront/behind* relation on any partially or fully expanded complex associated with an HCC composed of acyclic cell complexes. Thus, an HCC can be ordered back-to-front at any level of resolution.

Further developments of the work presented in this paper concern the implementation of a hierarchical simplicial complex in three dimensions for describing $3\frac{1}{2}$D surfaces at increasingly higher level of resolution, and the development of algorithms for isosurface extraction and direct volume rendering from the resulting model.

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