On Minimum and Maximum Visibility Problem

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Abstract

Given a polygon, the maximum visibility problem (**MaxVP**) asks for locating a point inside the polygon from which the visible area is maximized. Minimum visibility problem (**MinVP**) is defined similarly. Under standard visibility, we show that MinVP can be solved in $O(n^3)$ time for simple orthogonal polygons and in $O(n^3 \log n)$ time for orthogonal polygons with holes. Under stair-case visibility, we report following results: (a) We present an $O(n^3)$ time algorithm to solve MinVP and MaxVP for simple orthogonal polygons. The time complexity increases to $O(n^3 \log n)$ for orthogonal polygons with holes. (b) We show that for class 3 orthogonal polygons, solutions for MinVP and MaxVP lie on the boundary and present an $O(n^2)$ time algorithm to locate them.

1 Introduction

Study of the visibility properties of polygons have attracted the interest of many researchers in recent years. Under the standard definition of visibility, two points inside a polygon are said to be **visible** if the line segment connecting them does not intersect with the exterior of the polygon. An important visibility problem is the placement of the minimum number of point guards inside a polygonal gallery such that each point inside the polygon is visible to some guard. This problem is known as the "Art Gallery Problem" in the computational geometry literature [O'R87, Sh92, To88]. The general problem of placing the minimum number of point guards inside a polygonal gallery, with or without holes, is known to be NP-hard [LL86, Ag84]. An interesting visibility problem closely related to the art gallery problem is the computation of the visible area from a given **view point**. The area visible from a given view point q in the presence of obstacles is called the **visibility polygon** from q. Linear time algorithms for computing visibility polygon from a point inside a simple polygon are reported in [EA81, L83]. When the polygon contains holes visibility polygon from a point can be computed in $O(n \log n)$ time [As85], where n is the total number of vertices of the polygon.

In this paper, we consider problems related to the placement of point guards inside a polygon that maximizes/minimizes visibility. The maximum visibility problem (**MaxVP**) asks for locating a point inside a polygon from which the visible area is maximized. The minimum visibility problem (**MinVP**) is defined similarly. These kind of problems were introduced by Ntafos and Tsoukalas in [NT91], which contains an approximation algorithm of $O(n^4 \log b \log n)$ time and $O(n^2)$ space to b bits of accuracy. Under standard visibility, we present an exact algorithm to obtain a boundary point solution for MinVP. The algorithm runs in $O(n^3)$ time for simple orthogonal polygons and in $O(n^3 \log n)$ time for orthogonal polygons with holes. Under stair-case visibility, we report following results: (a) We present an $O(n^3)$ time algorithm to solve MinVP and MaxVP for simple orthogonal polygons; the time complexity increases to $O(n^3 \log n)$ for orthogonal polygons with holes. (b) We show that for class 3 orthogonal polygons, solutions for MinVP and MaxVP lie on the boundary and present an $O(n^2)$ time algorithm to locate them. Due to space limitation proofs of theorems and lemmas are omitted.

2 Algorithm Under Standard Visibility

In this section, we develop an algorithm to obtain a boundary point solution for MinVP for orthogonal polygons, under the standard definition of visibility. We partition the polygon into simpler parts by adapting the partitioning scheme used in [NT91] as follows: If a reflex vertex is visible to another vertex (reflex or non-reflex), we connect them by a straight line segment and extend it on both sides until it hits the boundary. Figure 1 shows an orthogonal polygon partitioned in this way. We use the term **reflex partitioning** to indicate such a partitioning. The visibility polygons from points on an edge induced by the reflex partitioning share some common properties. This is stated in the following lemma.

Lemma 1 [NT91]: The visibility polygons for points along an edge induced by the reflex partitioning contain the same set of vertices of the polygon.

We denote the line segment with end points a and b as (a, b). Consider a boundary edge $e = (a, b)$ induced by the reflex partitioning. Without loss of generality we assume that edge e is a horizontal edge and end point a is to the left of end point b (see Figure 2). Consider a triangulation of the visibility polygon V_x of a view point x on edge e. We next consider what happens to the triangles of the visibility polygon V_x as view point x moves from a to b along edge e. As x moves, some triangles remain the same and other change. We refer to the triangles that are completely visible from all points on edge e as **fixed triangles** and those that are partially visible from an interior point as **variable triangles**. In Figure 2, variable triangles are shaded. A variable triangle is completely visible from either point a or point b but is only partially visible from an interior point on e. Variable triangles that are completely invisible when x lies on point a become more and more visible as x moves from a towards b, and we call such triangles as **positive triangles**. Similarly, the variable triangles that are completely visible when x lies on a but become progressively invisible as x moves from a towards b are called **negative triangles**.

The base of each variable triangle lies on the boundary of the polygon. We refer to variable triangles with base lying on horizontal edges as **horizontal triangles** and those with base lying on vertical edges as **vertical triangles**.

Lemma 2: The area of a positive horizontal triangle as a function of distance l_1 between view point x and end point a is a straight line, i.e., $A_h^+(l_1) = K_1 l_1$, where K_1 is a positive constant. Similarly, the area of a negative horizontal triangle can be written as $A_h^-(l_1) = K_2 - K_3 l_1$, where K_2 and K_3 are positive constants.

Lemma 3: The area of a positive vertical triangle as a function of distance l_1 between view point x and end point a can be written as: $A_v^+(l_1) = C_1 - \frac{C_2}{l_1 + C_3}$, where C_1, C_2, and C_3 are positive constants. Similarly, the area of a negative vertical triangle as a function of l_1 can be written as: $A_v^-(l_1) = C_1 - \frac{C_2}{l_1 - C_3}$.

Lemma 4: The boundary point solution for MinVP for orthogonal polygons is given by one of the end points of the edges induced by the reflex partitioning.

Theorem 1: The boundary point solution for MinVP can be computed in $O(n^3)$ time for simple orthogonal polygons and in $O(n^3 \log n)$ time for orthogonal polygons with holes.

For exploring the visibility properties of orthogonal polygons, notion of stair-case path and stair-case visibility have been useful [RC87, MRS90, GKN92]. A **stair-case path** consist of horizontal and vertical line segments such that its intersection with any horizontal or vertical line is at most one line segment. In other words, a stair-case path is monotone along both x-axis and y-axis directions. Two points inside an orthogonal polygon are visible under **stair-case visibility** if they can be connected by a stair-case path without intersecting its exterior. Stair-case visibility concepts have been used to decompose orthogonal polygons into simpler polygons [MRS90, RC87]. If we traverse the boundary of an orthogonal polygon P in the clockwise direction, keeping the interior to the right, then at vertex of the polygon we either turn 90^0 right (outside corner) or 90^0 left (inside corner). A **dent** is an edge of the polygon whose both end points are inside corners. The direction of dent traversing gives it orientations which we indicate as **N, S, E** and **W** dents. A line segment inside a polygon with its end points at the boundary is called a **chord** of the polygon. A chord obtained by extending dent edges is called a **dent-chord**.

Consider the partitioning of a simple orthogonal polygon by dent-chords. We refer to such a partitioning as the **dent partitioning**. It is easy to see that the visibility polygons within the same region induced by the dent-partitioning are identical. The stair-case visibility polygon from a point inside a simple orthogonal polygon can be computed from the trapezoidizations of the polygon obtained by extending its horizontal/vertical edges in linear time. The detail of the algorithm is reported in [G93]. To solve the minimum and maximum visibility problem we compute stair-case visibility polygon for a point from each region, compute corresponding areas and report the ones with minimum and maximum area. Since there can be $O(n^2)$ regions in a dent partitioning, and since the area of a visibility polygon can be computed in linear time by using the triangulation algorithm given in [Ch90], we can state the following theorem.

Theorem 2: Under stair-case visibility, the solutions for MaxVP and MinVP can be obtained in $O(n^3)$ time for simple orthogonal polygons.

When the polygon has holes the time complexity of computing stair-case visibility polygon becomes $O(n \log n)$ [G93] and hence the time complexity for solving MinVP and MaxVP increases by a factor of $O(\log n)$.

It is tempting to explore whether the solution for MinVP and MaxVP lie on the boundary or not. Examples can be constructed where only non-boundary points can maximize/minimize visibility. Similar examples can be constructed where only boundary equivalent points can minimize/maximize visibility. A related issue would then be to identify class of orthogonal polygons for which the solution lies on the boundary. Reckhow and Culberson [RC87] have demonstrated the usefulness of classifying polygons in term of dent orientations. A simple orthogonal polygon is said to be a **class k** polygon if it has exactly k dent orientations. Under this scheme, the **class 4** polygons are precisely the set of all simple orthogonal polygon and the **class 3** polygons are simple orthogonal polygons having dent orientation only in three directions. Some difficult problems on orthogonal problems can be solved relatively easily when the polygon is restricted to be a class 3 polygon. For example, Motwani et al [MRS90] have shown that the minimum cover of a class 3 polygon by stair-case star polygons can be found in $O(n^3)$ time.

Lemma 5: Under stair-case visibility, solution for MaxVP for class 3 orthogonal polygons can be found on the boundary.

Lemma 6: Under stair-case visibility, solution for MinVP for class 3 orthogonal polygons can be found on the boundary.

Theorem 3: Under stair-case visibility, solution for MinVP and MaxVP for class 3 polygons can be computed in $O(n^2)$ time.

3 Discussions

We proved that, under standard visibility, the boundary point solution for MinVP for orthogonal polygons is given by one of the end points of the segments induced by the reflex partitioning and presented an algorithm to locate the solution. The time complexity of the algorithm is $O(n^3)$ for simple orthogonal polygons and $O(n^3 \log n)$ for orthogonal polygons with holes. Under stair-case visibility, we showed that MinVP and MaxVP can be solved in $O(n^3)$ time for simple orthogonal polygons and in $O(n^3 \log n)$ time for orthogonal polygons with holes. For class 3 orthogonal polygons under stair-case visibility, we proved that solutions for MaxVP and MinVP lie on the boundary and presented an $O(n^2)$ time algorithm to locate them. A natural extension would be to pursue for more efficient algorithms. The structural relationship between the visibility polygon of adjacent vertices induced by dent partitioning might give insight to develop faster algorithms.

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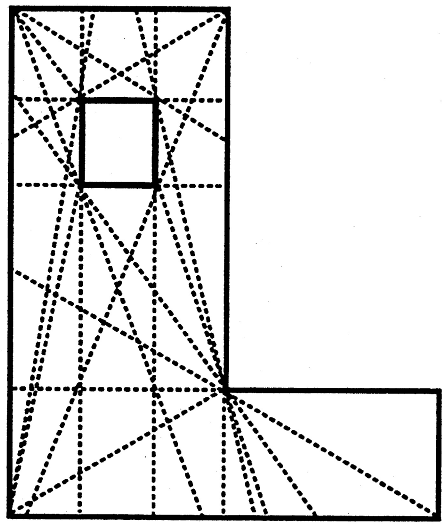

Figure 1: Illustrating Reflex Partitioning.

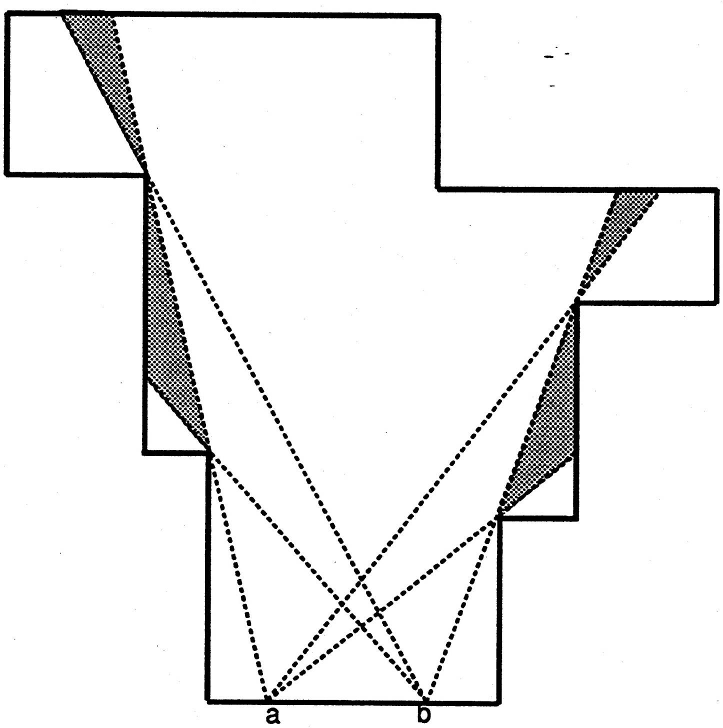

Figure 2: Illustrating Variable Triangles (Shaded).