Looking Through a Window

(preliminary version)

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Abstract.

In this paper we describe algorithms for optimizing the angle of vision through the gap between two objects, when the viewpoint belongs to a given trajectory.

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1. Introduction

We begin with some definitions and notations. The convex hull of any set S will be denoted $conv(S)$. If S is compact and connected, and X is a point not in $conv(S)$, the ray emanating from X, tangent to S, and having S to its right (resp. left) will be called *the left tangent* (resp. *the right tangent*) and denoted $l_X(S)$ (resp. $r_X(S)$), the reference to S omitted whenever possible.

Let $P, Q_1, \ldots, Q_k$ be disjoint geometrical objects in the plane (all supposed to be compact and connected), and let T be a set of points, usually some kind of curve, that we will call *the trajectory* (refer to Figure 1). We say that P *is seen* from a point $X \in T$, $X \notin conv(P)$, when no point of $T \cup Q_1 \cup \cdots \cup Q_k$ belongs to the interior of $conv(P \cup X) - conv(P)$. More intuitively: no visual ray from X to P is intercepted by any "obstacle" Q_i or by T itself.

We consider the following two general problems:

Problem 1. Let T_P be the set of points of T from which P can be seen. *Is T_P empty? If $T_P \neq \emptyset$ from which point of T_P will the object P be seen with maximum angle?*

Problem 2. Let us assume that $k = 2$ (refer to Figure 2) and that $conv(Q_1 \cup Q_2)$ intersects neither P nor T . Let T^P be the set of all points $X \in T$ admitting rays $r_{1,X}$, $r_{2,X}$, respectively tangent to Q_1 and Q_2, both emanating from X, such that: a) Q_1 and Q_2 are linearly separated by both $r_{1,X}$ and $r_{2,X}$; b) The convex angle $\Gamma(X)$ defined by $r_{1,X}$ and $r_{2,X}$ contains P in its interior and $Q_1 \cup Q_2$ in its exterior; c) No point of T belongs to the interior of $\Gamma(X)$. More intuitively, T^P is the set of points of T such that the angle of vision through the gap between Q_1 and Q_2 contains P. *Is T^P empty? If $T^P \neq \emptyset$ what point $X \in T^P$ provides a maximum $\Gamma(X)$?*

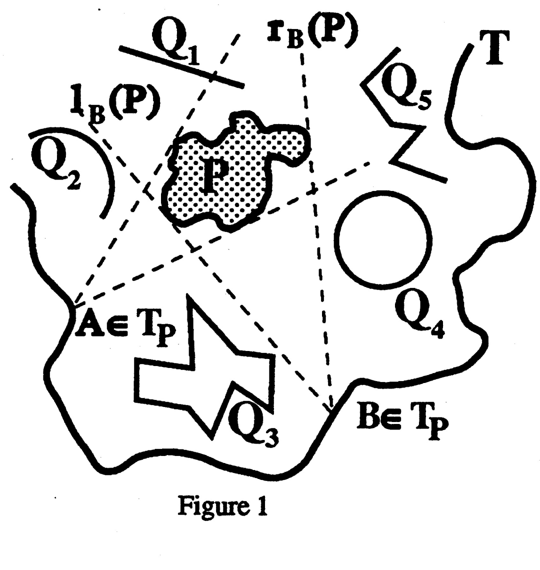

Figure 1

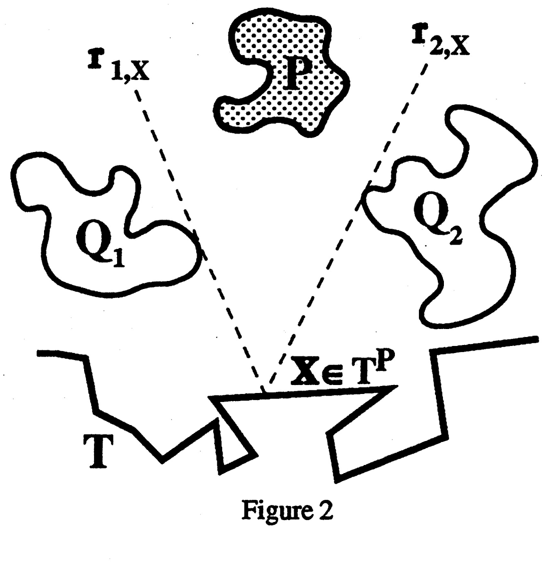

Figure 2

The complexity of these problems depends on:
The trajectory (the set *from which we look*);
The object P (the set *at which we look*);
The Q_i's (the obstacles *through which we look*).

Problem 1 and Problem 2 are very similar but the crucial question differs. In the first problem, once T_P is obtained we concentrate on looking at P. Here the main question is for a given trajectory (T_P) to maximize the angle from whose apex a given object is seen. Some cases of this problem are described in [1] and [2]. In [3,4] a fixed-size angle is given, but the apex is free on the plane, the goal being to locate that apex as near as possible to the object. In this paper we focus our attention on Problem 2: once T^P is obtained, we can forget P, and we must optimize the vision through a gap from a given trajectory (T^P).

2. Previous results: looking through a segment

To *look through* a segment AB (or through the points A and B) is the same thing as to *look at* the segment AB. As a consequence the basic tool for the two types of problems described in preceeding paragraphs is the same, as we shall see below. We briefly review in this section some results about looking at a segment. Omitted proofs are mostly from elementary geometry. Computational issues are described in [1,2]

The set of points to the left (resp. right) of the ray AB will be denoted $L(AB)$ (resp. $R(AB)$). For simplicity we will restrict our attention to one of these half planes.

The locus of points of $L(AB)$ that see AB with a given angle α is a circular arc C with extremes A and B. In $L(AB)$, from points inside the region R bounded by C and AB, the segment AB is seen with an angle greater than α, the angle being lesser than α from points outside R. Inside $L(AB)$, when a point X moves away from R along a ray starting at a point of C the magnitude of the angle AXB is a strictly decreasing function.

These facts give us a strategy for finding the points of a given geometric object P that see AB with the largest angle (Fig. 3): "inflate" a circular arc over AB until P is touched for the first time (when working in both half planes one has to deal with two such arcs). We state this more precisely for future reference:

Lemma 1. *If P is a circle, a straight line, a ray or a segment, the point of P (occasionally two points) that see a given segment AB with maximum angle can be found in time $O(1)$.*

Lemma 2. *If P is the union of n segments (a n-polygonal, a general n-polygon), the points of P that see a given segment AB with maximum angle can be found in optimal time $O(n)$.*

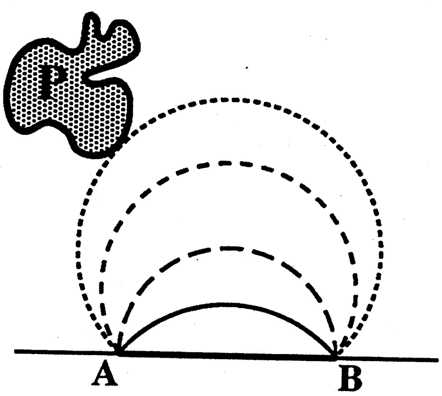

Figure 3

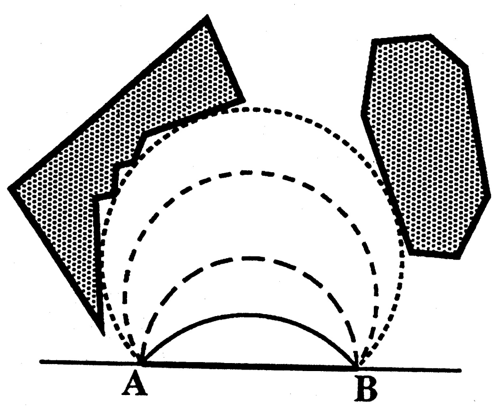

Figure 4

Lemma 3. *Let AB be a segment and let P be the portion inside L(AB) of a straight line, a segment, a ray, or a polygonal convex towards AB. The magnitude of the angle of vision of the segment AB from points of P is a max-unimodal function (unimodal with maximum).*

Lemma 4. *Let P be a convex polygon with n sides and let AB be a segment in the exterior of P. The point of P (occasionally two points) that see AB with maximum angle can be found in* $O(\lg n)$ *time.*

The bound in Lemma 2 can be achieved by iterating the optimization on every segment; the optimality comes from the output size (Fig. 4). By Lemma 3 this argument is no longer valid for convex polygons, binary search giving Lemma 4.

3. Looking through two polygons

Let Q_1, Q_2 be disjoint polygons (or polygonals) with a total of n vertices and let T be a trajectory not intersecting $conv(Q_1 \cup Q_2)$. Let $X \in T$ be a point not belonging to any line defined by the prolongation of a side of $conv(Q_1)$ or $conv(Q_2)$. If X can see through the gap between Q_1 and Q_2, the tangent rays $r_{1,X}$, $r_{2,X}$ defining such angle will touch Q_1 and Q_2 in single points A_1 and A_2. In some neighborhood of X all the points of T can see through the gap between Q_1 and Q_2 exactly through the segment A_1A_2. This fact suggests the following general algorithm:

Algorithm THROUGHPOLYGONS

Input. Two disjoint polygons (or polygonals) Q_1, Q_2 with a total of n vertices , and a trajectory T not intersecting $conv(Q_1 \cup Q_2)$.

Output. The angle of maximum vision from points of T through the gap between Q_1 and Q_2, and the list of points where it is reached (when existing).

1 Find $C_1 = conv(Q_1)$ and $C_2 = conv(Q_2)$. If $C_1 \cap C_2 \neq \emptyset$ return the message "No gap" and exit.

2 Find the inner common tangents to C_1 and C_2 and determine the subset T' of T of the points that can see through C_1, C_2. If $T' = \emptyset$ return the message "No visible gap from T" and exit.

3 Decompose T' in subzones $T_1, \ldots, T_k$ such that from every T_i the vision through C_1, C_2 is the vision through a fixed segment with an extreme in every polygon.

4 Using the techniques from the preceeding section, optimize the vision from every T_i, maintaining the value α of the maximum obtained angle and the list L of the points where it is reached.

5 Return α and L, and exit.

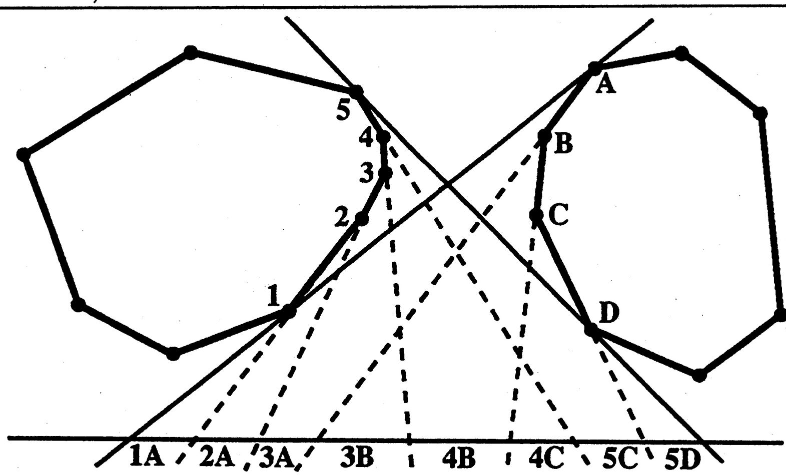

Figure 5

Details and running time depend on the specific types of geometric objects. For the sake of clarity and future convenience, we will focus on the case T is an straight line (Fig. 5). In this situation Steps 1 and 2 can be completed in time $O(n)$. In Step 3, we extend the sides of C_1 and C_2, intersect with T', and sort the obtained points. All of this can be accomplished in time $O(n)$, because we are merging two sorted lists. We obtain $O(n)$ subsegments of T; in each one the optimization is done in constant time. So we have:

Proposition 1. *Let Q_1, Q_2 be disjoint polygons or polygonals with a total of n vertices, and let s be a straight line or segment not intersecting $conv(Q_1 \cup Q_2)$. The points of s with maximum angular vision through the gap between Q_1 and Q_2, and the magnitude of such an angle, can be obtained in $O(n)$ time.*

With the same technique, but different costs for computing intersections and in Step 4, we obtain the following results (for details, see [2]):

Proposition 2. *Let Q_1, Q_2 be disjoint polygons or polygonals, with a total of n vertices, and let P a polygon or polygonal with m vertices. The points of P with maximum angular vision through the gap between Q_1 and Q_2, and the magnitude of such an angle, can be obtained in $O(nm)$ time.*

Proposition 3. *Let Q_1, Q_2 be disjoint polygons or polygonals, with a total of n vertices, and let P be a convex m-polygon containing Q_1 and Q_2. The points of P with maximum angular vision through the gap between Q_1 and Q_2, and the magnitude of such an angle, can be obtained in $O(m + n \lg m)$ time.*

Proposition 4. *Let Q_1, Q_2 be disjoint polygons or polygonals, with a total of n vertices, and let P be a convex m-polygon in the exterior of $conv(Q_1 \cup Q_2)$. The points of P with maximum angular vision through the gap between Q_1 and Q_2, and the magnitude of such an angle, can be obtained in $O(n \lg m)$ time.*

Some special cases can be handled with *ad hoc* techniques. For example, in the situation of Proposition 1, if Q_1 and Q_2 are convex, the first two steps can be done in logarithmic time. But to beat the $O(n)$ bound we need the following lemma:

Lemma 5. *Let $f_i : [a, b] \longrightarrow \mathbb{R}$ $(i = 1, \ldots, k)$ max-unimodal continuous functions, and let $f : [x_0, x_k] \longrightarrow \mathbb{R}$ be piecewise defined in such a way that*
(1) $a < x_0 < \cdots < x_k < b$;
(2) $f \,\big|\, [x_{i-1}, x_i] = f_i \quad i = 1, \ldots, k$;
(3) $\exists \epsilon \in \mathbb{R}^+$ such that $(x - x_i)(f_{i+1}(x) - f_i(x)) \le 0$ in $[x_i - \epsilon, x_i + \epsilon] \quad i = 1, \ldots, k-1$.
Then f is max-unimodal in $[x_0, x_k]$.

Proof. At x_i the function f transforms from f_i to f_{i+1}. Condition (3) means that f_{i+1} crosses f_i at x_i in "top-down" manner: $f_{i+1} \ge f_i$ immediately to the left of x_i and $f_{i+1} \le f_i$ immediately to the right of x_i, so excluding the possibility that f has a local minimum at x_i. As the f_j are max-unimodal, f is continuous and without local minimum in (x_0, x_k), and so f is max-unimodal. Q.E.D.

Now let's consider two convex polygons C_1 and C_2, as in Fig. 5. Without lost of generality we can suppose that they are both in the half plane $y > 0$ and that the common inner tangents determine a segment s in the x-axis for looking through the gap. The angular vision from s through every segment with one extreme in each polygon ("inside the gap") is a max-unimodal function, by Lemma 3. When a point moves on s from left to right, such functions are piecewise concatenated in the manner of Lemma 5, so we have the following

Lemma 6. *Let C_1, C_2 be convex polygons and let s be a segment, a ray or a straight line leaving both polygons in the same associated half plane. The angular vision from s through the gap between C_1 and C_2 is a max-unimodal function.*

Now we can take the sides halving the chains of the polygons facing each other, extend them, and intersect with s. The values of angular vision from the obtained points allow us to discard at least half of one chain. The process is repeated a logarithmic number of times, each step having logarithmic cost. So we have

Proposition 5. *Let C_1, C_2 be convex polygons with a total of n vertices, and let s be a segment, a ray or a straight line not intersecting the interior of $conv(C_1 \cup C_2)$. The maximum angle of vision from s through the gap between C_1 and C_2, and the points where it is reached, can be obtained in $O(\lg^2 n)$ time.*

The argumentation preceeding the last lemma and proposition can be rephrased *mutatis mutandis* if instead of a segment we take a polygonal chain convex towards the gap. This gives us the last result in this section:

Proposition 6. *Let C_1, C_2 be convex polygons with a total of n vertices, and let be D a convex polygon with m vertices, not intersecting $conv(C_1 \cup C_2)$. The maximum angle of vision from D through the gap between C_1 and C_2, and the points where it is reached, can be obtained in $O((\lg m + \lg n) \lg n)$ time.*

4. A glance at an application

We conclude this paper by sketching how the former techniques apply to a problem arising in the context of [5], where *unoriented* Θ*-maxima*, a generalization of classical maxima of a set of points (or vectors), were introduced. A detailed description, and additional related results will appear in [6].

Let S be a set of n points in the plane, and let $P \notin S$ be a point inside $conv(S)$. When we look from P, the maximum angle α free of points of S (that will be considered as obstacles) can be clearly obtained in $O(n \lg n)$ time. In the reverse situation we are free to move in the plane, and we look at P with an angle as large as possible, free of any obstacle; the set of values is upper bounded by α and we can approximate indefinitely that value by approaching P (in a suitable direction). But what if we must remain outside S, say in the exterior of $conv(S)$?

The first thing we observe is that it suffices to optimize the vision from points on the boundary of $conv(S)$: if from a point Q, strictly outside of $conv(S)$, P is seen within some angle free of obstacles, from the point where the segment PQ intersects $conv(S)$ a greater value is obtained.

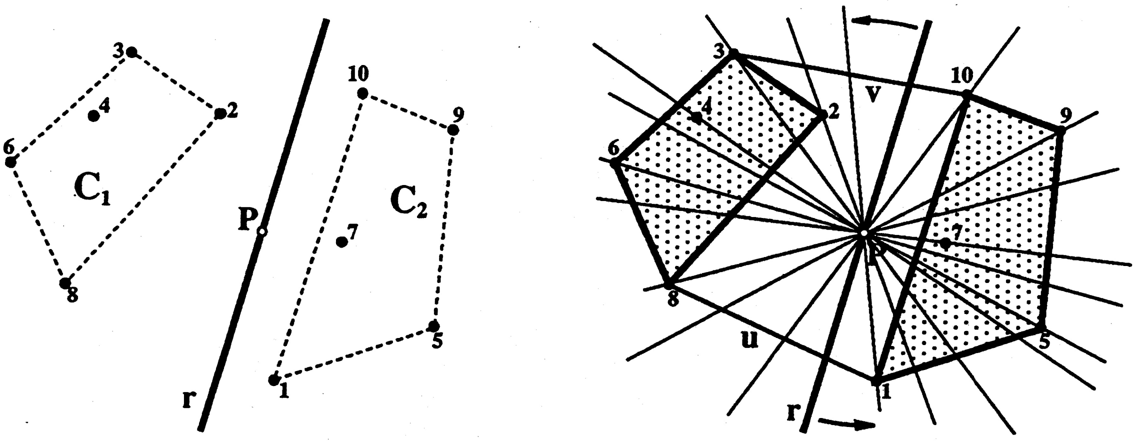

Figure 6

Now we take any line r through P, not containing any point of S, that separates in two subsets S_1 and S_2, having convex hulls C_1 and C_2 respectively (Fig. 6). Let u, v be the segments of common outer tangents bridging C_1 and C_2 (that is, the sides of $conv(S)$ crossed by r). After optimizing the vision from u and v through the gap between C_1 and C_2, we rotate r until a first point is surpassed, then S_1, S_2, C_1, C_2, u and v are updated and we iterate the process. As the optimization is the dominant step, Proposition 5 gives us the following

Proposition 7. *Let S be a set of n points in the plane, and let $P \notin S$ be a point inside conv(S). The points of the plane that can see P with maximum angle, free of any point of S, and the value of such an angle, can be obtained in $O(n \lg^2 n)$ time.*

This strategy can be easily adapted for different definitions of *outside* S, or to the case we are looking from trajectories or regions.

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