Grazing Inside a Convex Polygon

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Abstract

Given a rope whose one end is fixed at a point on the boundary of a convex polygonal obstacle called the *axis point*, the internal grazing area is defined as the set of points internal to the obstacle that can be reached by the rope. A $O(n^2)$ time algorithm to locate the boundary point to tie the rope that corresponds to the minimum internal grazing area is presented. This problem has many engineering applications in environmental studies such as pollution control and toxic spills.

1 Introduction

Problems dealing with visibility of polygons have been studied by many researchers in computational geometry [1-3]. A variation of the visibility problem called the External Grazing Area Problem (or **EGAP**) asks for the following: Given a convex polygonal obstacle Q of n vertices and a rope of fixed length L, find a boundary point (the **axis point**) such that when one end of the rope is fixed at the axis point, the area that can be swept (grazed) by the rope outside the polygon is a maximum. In [4], it is established that the axis point yielding the maximum grazing area is given by one of the vertices of the polygon which results in an $O(n^2)$ algorithm for locating it. Two types of external grazing areas were identified; (a) simple grazing area and (b) general grazing area. An optimal $O(n)$ time algorithm to solve **EGAP** for the case of a simple grazing area is obtained in [4].

A problem closely related to the **EGAP** is the minimum internal grazing area problem (**MIGAP**). MIGAP problem asks for locating the axis point on the boundary of the polygon so that the total grazing area swept by the rope internal to the convex polygon, is minimized. In this paper, it is proved that the solution to MIGAP lies on one of the $5n$ bondary points, n of which are the original vertices and the rest are created using simple rules. A discussion of these rules is presented. An $O(n^2)$ algorithm is developed to compute the location of the axis point.

2 Preliminaries

If the length of the rope is smaller than the smallest edge of the polygon, then the solution to MIGAP lies at the vertex of the smallest internal angle which is a trivial observation (Figure 1a). On the other hand, if the length of the rope is larger than the largest diameter of the polygon, then all the boundary points give rise to the same solution to MIGAP which is the area of the polygon (Figure 1b). But, for the case when the rope is smaller than the largest diameter and larger than the length of the smallest edge, the solution to MIGAP is nontrivial (Figure 1c). We, therefore, consider the case of Figure 1c.

Let us consider the convex polygon shown in Figure 2. As the axis point moves along the boundary of the polygon, the internal grazed area changes as a function of the position of the axis point along the boundary of the polygon. Let us call this function *grazing area function* and note that it is continuous. But the first derivative of the function is possibly discontinuous at some boundary points. We introduce new vertices at the locations of the axis points corresponding to these possible discontinuities in the slope of the function and call them steiner vertices. The new set of vertices consisting of the original and the steiner vertices are such that all order derivatives of *grazing area function* are continuous between any two adjacent ones.

The discontinuous changes in the slope of the *grazing area function* can possibly occur in the following three situations; (i) when the axis point crosses over from one edge to an adjacent edge through an original vertex (Type I; Figure 3a), (ii) when the circular boundary of the *grazing area* crosses over an edge from the inside to the outside of the polygon (Type II; Figure 3b), and (iii) when the circular boundary of the *grazing area* crosses over from one edge to an adjacent one through a vertex (Type III; Figure 3c). Type II vertices can be obtained by drawing lines parallel to the edges of the polygon at a distance of the length of the rope and identifying the points of intersection of the parallel line with the edges of the polygon. There will be 2n such intersections. These points of intersections are the type II vertices. Some of these axis points will correspond to the circular sector crossing the extended edges of the polygon. These axis points should be excluded. Type III vertices can be obtained by drawing circles of radius r and with the center at each original vertex and identifying the pairs of intersections of the circle with the edges of the polygon. When the axis point is at these intersection points, the circle will cross over from one edge to an adjacent one through a particular vertex. There will be 2n such intersections, but some them can be excluded as they intersect the extended edges of the polygon. Thus, the number of possible discontinuities in the slope of the *grazing area function* and the corresponding number of axis points that need to be tested for local minimum in the *grazing area function* is utmost $5n$. Let us index the new set of vertices in the clockwise order in which they appear and call them vertices.

Figure 2 shows that the axis point is at X and the circle is intersecting the polygon at O_1, O_2, O_3 and O_4. Let us consider the variation of the area as a function of the position of the axis point along an edge between two adjacent vertices. Without loss of generality, let us assume that the edge on which the axis point is located is the x axis.

The boundary of the grazed area consists of circular arcs and boundary chains of the polygon. The circular arcs along with two radial lines originating from the axis point form a circular sector. Similarly, two radial lines originating from the axis point along with the boundary chains of the polygon form polygonal sectors. For example, for the polygonal object shown in Figure 2, when the axis point is on the x axis at X, the grazed area consists of circular sectors, XO_1O_2 and XO_3O_4 and convex polygonal sectors, XAG, XO_2EO_3 and XO_4CB. Let us call the area of the i^{th} circular sectors in the clockwise direction $A^i_C(x)$. Similarly, let us call the i^{th} polygonal sectors $A^i_P(x)$. With these definitions, the *grazing area function*, $A(x)$ can be written as:

$$A(x) = \sum_{i=1}^{i=n_c} A^i_C(x) + \sum_{i=1}^{i=n_p} A^i_P(x) \tag{1}$$

where x is the axis point location on the boundary of the polygon. n_c and n_p are the total numbers of circular and polygonal sectors, respectively.

Lemma 1: The area of the circular sector along with the two neighboring right triangular polygonal sectors (formed by the radial edge of the circular sector, part of the edge which the circular arc cuts and the perpendicular line from the axis point to that edge) one on each side of the sector, as a function of the axis point along the boundary, has its second derivative always negative.

Proof: Consider the circular sector, XO_1O_2, shown in Figure 4. It is noted that Figure 4 also shows the partial neighboring polygonal sectors, XO_1A and O_2P_2X. The area of the circular sector, XO_1O_2, denoted as $A^i_C(x)$ is given by:

$$A^i_C(x) = \beta r^2 \tag{2}$$

where r is the radius of the rope, and β is given by:

$$\beta = \left((2n-4)\frac{\pi}{2} - \pi - cos^{-1}\left(\frac{h_i}{r}\right) - cos^{-1}\left(\frac{h_{i+1}}{r}\right)\right) \tag{3}$$

where h_i and h_{i+1} are the perpendicular distances from the the axis point to the the edges of the polygon which form the circular sector, (i.e.), GA and FE respectively and are shown in Figure 4. The area $P_2O_2O_1AX$ includes the circular sector O_1XO_2 and the two neighboring right triangular sectors, O_1XA and O_2XP_2. Writing it as a function of h_i and h_{i+1} and denoting it as $A^i_{CT}(h_i, h_{i+1})$,

$$\begin{aligned} A^i_{CT}(h_i, h_{i+1}) &= Area\ O_1P_1X - Area\ AP_1X + Area\ O_1O_2X + Area\ O_2P_2X \\ &= h_i^2 tan(\phi_i) - h_i^2 tan(\alpha_i) + \beta r^2 + h_{i+1}^2 tan(\phi_{i+1}) \\ &= h_i\sqrt{r^2 - h_i^2} - h_i^2 tan(\alpha_i) + \beta r^2 + h_{i+1}\sqrt{r^2 - h_{i+1}^2} \end{aligned} \tag{4}$$

Between any two adjacent vertices, it can be proved by similar triangles arguments that h_i and h_{i+1} are linear functions of x which is the location of the axis point along the boundary. Thus, it is obvious that $K_i dx = dh_i$ and $K_{i+1} dx = dh_{i+1}$, where K_i and K_{i+1} are real constants. Using the above linear relationship and Equation 4, the second derivative, $\frac{d^2 A^i_{CT}(x)}{dx^2}$, can be shown to be:

$$\frac{d^2 A^i_{CT}(x)}{dx^2} = -K_i^2\left(\frac{2h_i(x)}{\sqrt{r^2 - h_i(x)^2}} + 2tan(\alpha_i)\right) - K_2^2\left(\frac{2h_{i+1}(x)}{\sqrt{r^2 - h_{i+1}(x)^2}}\right) < 0 \tag{5}$$

due to the fact that K_i and K_{i+1} are real and the terms under square root are always positive as $r > h_i$ and $r > h_{i+1}$, since r is the length of the hypotenuses of the right triangles,XP_1O_1 and XO_2P_2. Hence the lemma. □

It is noted that the right triangular sectors already considered in the above analysis should be discounted in the following analysis of the convex polygonal sectors.

Lemma 2: The area of a polygonal sector as a function of the boundary location of the axis point has its second derivative always negative.

Proof: Consider the i^{th} polygonal sector P_2EM_2X as shown in Figure 5. Area of the sector P_2EM_2X as a function of x can be written as:

$$\begin{aligned} A^i_P(x) &= A^i_{P_1}(x) + A^i_{P_2}(x) \\ &= Area\ M_1EX + Area\ M_2EX \\ &= h_{i+1}(x)\sqrt{l^2 - h_{i+1}(x)^2} + h_{i+2}(x)\sqrt{l^2 - h_{i+2}(x)^2} \end{aligned} \tag{6}$$

where $l^2 = (x - x_i)^2 + y_i^2$. l, h_{i+1} and h_{i+2} are the perpendicular distances from the axis point to the two bounding edges of the polygonal sectors, EF and ED and are shown in Figure 5. (x_i, y_i) are the coordinates of the point E. There can be additional polygonal area sandwiched between these two triangular sectors, but their area remains constant as axis point moves along the edge till one of the triangular area goes to zero or reaches a maximum value. The axis point corresponding to such a situation is a steiner vertex. As the additional sandwiched polygonal area does not vary with x between the two adjacent vertices under consideration, it does not affect the variational problem and therefore, is not considered. Therefore, functional arguments based on Equation 6 are made to show that the second derivative of the *grazing area function* is negative. Consider Figure 5. Let us denote the point of intersection of EO_2 with x axis as the temporary origin. As $x \longrightarrow 0$, $h_{i+1} \longrightarrow 0$. Then $A^i_{P_1}(x)$ given by Equation 6 goes to zero. Similarly, when $l \longrightarrow h_{i+1}$, Area $A^i_{P_1}(x)$ given by Equation 6 goes to zero. Additionally, the $A^i_{P_1}(x)$ is always positive and has only two extrema. With these observations about the function $A^i_{P_1}(x)$, it is obvious that it should appear as shown in Figure 6 with one extrema within the interval $x \longrightarrow 0$ and $l \longrightarrow h_{i+1}$. Similar arguments can be posed for the function $A^i_{P_2}(x)$. From Figure 6 and Equation 6, it is obvious that $\frac{d^2 A^i_P(x)}{dx^2} < 0$. □

Any other possible circular sector or convex polygonal sectors is a simpler case of the ones that are discussed under **Lemma 1** and **Lemma 2.**

Theorem 1: The *grazing area function* between two adjacent vertices exhibits only a maximum.
Proof: From Lemmas 1 and 2, it follows that for every $A^i_C(x)$ and $A^i_P(x)$, the second derivatives w.r.t. x are negative. Moreover, notice that the grazing area function given by Equation 1 is a sum of functions given by Equations 4 and 5. Therefore, it is easily seen that the second derivative of the *grazing area function* is negative, when the axis point is between any two adjacent vertices. if there is an extremum for an axis point between two adjacent vertices, it can only be a maximum. In other words, the minimum of the grazing area function occurs at the vertices of the polygon. □

Theorem 2: The axis point yielding the minimum *grazing area function* can be computed in $O(n^2)$ time.
Proof: From **Theorem 1**, it is proved that the local minima of *grazing area function* are given by the vertices of the polygon. Therefore, it suffices to search only the values of the *grazing area function* corresponding to the vertices to obtain the global minimum. The grazed area corresponding to every vertex can be computed in $O(n)$ time as there are $O(n)$ intersections of the circular boundary of the grazed area with the convex polygon and hence there are $O(n)$ circular sectors and convex polygonal sectors. There are utmost $5n$ vertices to be checked and therefore, the global minimum of the *grazing area function* can be obtained in utmost $5n^2$ time. □.

3 Conclusion

We proved that a minima of the internal grazing area occurs when the axis-point is on the vertex of the polygon. There are utmost $5n$ vertices and the rules to obtain them are simple and are given. Based on these observations, an algorithm with utmost $5(n^2)$ time for computing the minimum internal grazing area is presented. It will be interesting to develop a subquadratic algorithm for MIGAP.

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4 References

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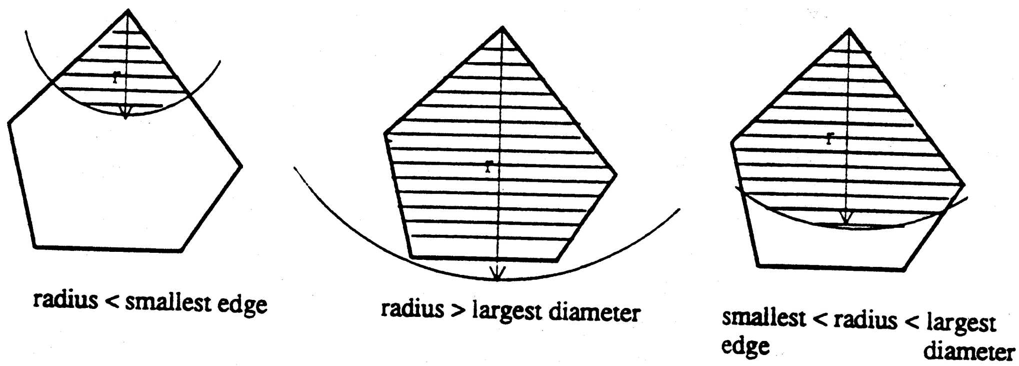

Figure 1. Two trivial and one nontrivial cases of a **MIGAP**

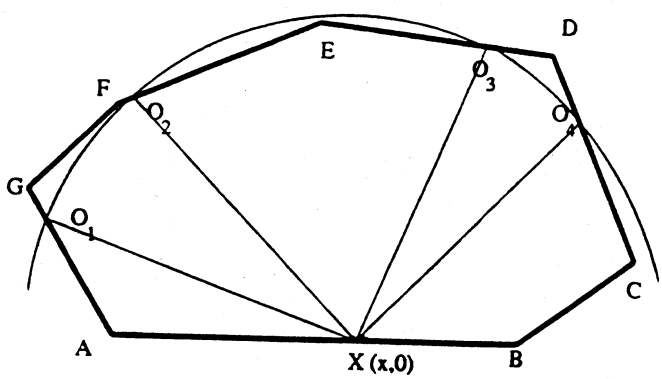

Figure 2. A convex polygon along with thr grazed area showing the intersections of the convex polygon with circular boundary of radius r and center $X(x,0)$ located on the boundary of the polygon.

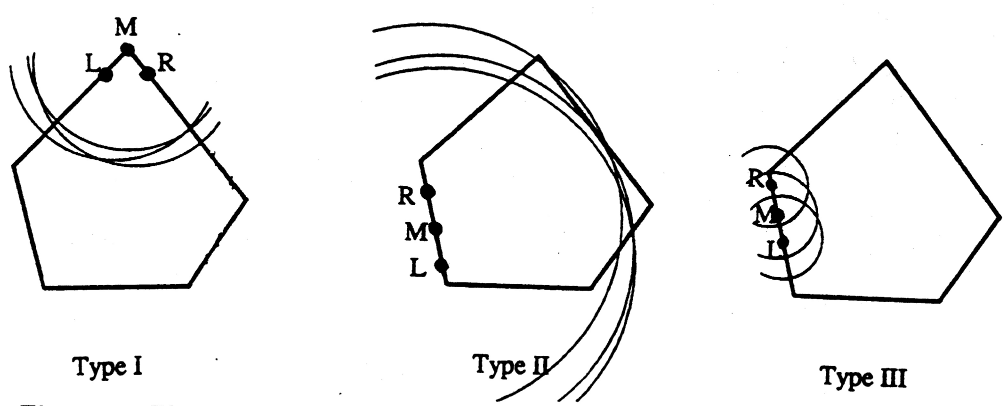

Figure 3. Three types of possible discontinuities in the slope of *grazing area function.* (i) axis point crossing over an original vertex (ii) circular boundary crossing over an edge (iii) circular boundary crossing over an original vertex.

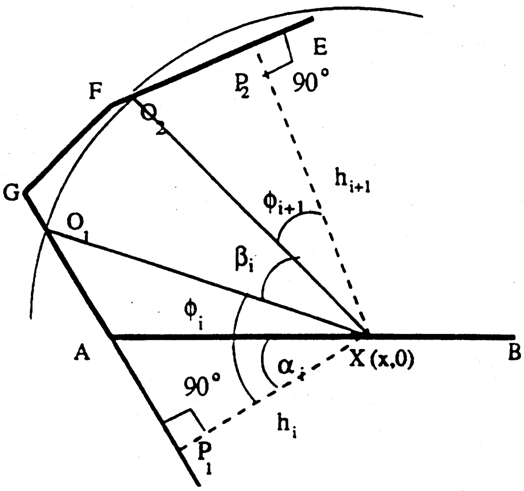

Figure 4. Circular sector along with its two neighboring right triangular sectors.

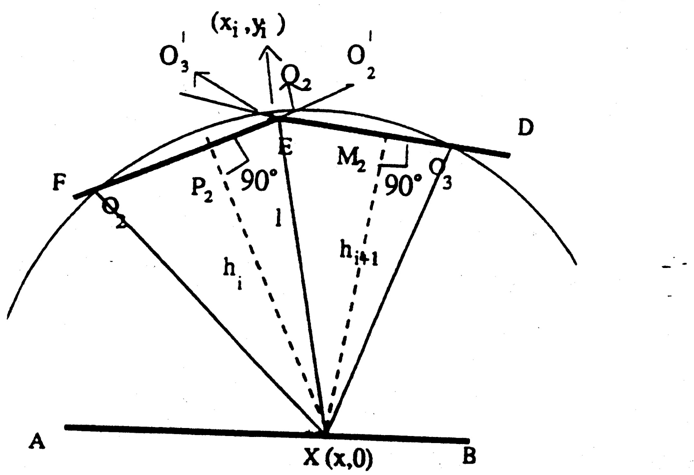

Figure 5. Polygonal sector. EO_2 and BA intersect at the origin $(0,0)$.

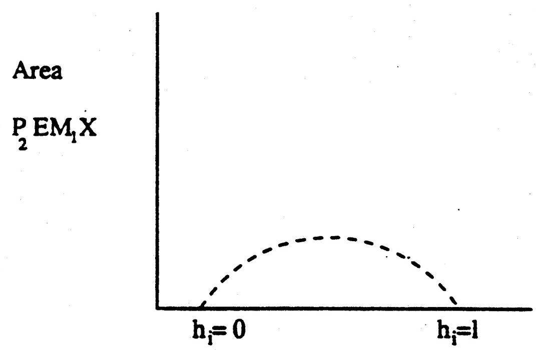

Figure 6. Grazing area function, $A_P^i(x)$, for a general polygonal segment as a function of position of the axis point along the boundary of the polygon.