Shortest Paths for a Two-Robot Rendez-Vous

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Abstract

In this paper, we consider an optimal motion planning problem for a pair of point robots in a planar environment with polygonal obstacles. We seek a pair of paths upon which the robots can travel to move from their initial positions to positions in which they are able to communicate. In order to communicate, the robots need to be visible to one another. We give efficient algorithms for minimizing either the sum or the maximum of the two path lengths.

1 Introduction

Imagine that two mobile robots are working simultaneously but separately on related tasks. Suppose that they must "rendez-vous" periodically to share information which one robot has acquired which may be useful to the other robot in carrying out its assigned task. Depending upon the model of communication used, the robots may need to travel to the same location to share information, or they may only have to move so that they are within a prescribed distance of one another or so that they are visible to one another.

We assume that the robots use a line-of-sight communication system. Therefore, we seek a pair of *rendez-vous paths* upon which the robots can travel to become visible to each other. Our goal is to find optimal solutions to two variations of this problem. In one, we wish to minimize the *sum* of the lengths of the rendez-vous paths; in the other, we wish to minimize the *maximum* of the path lengths.

Given the situation described above, our goal is to find a pair of paths for the robots upon which they can travel to reach a configuration (a pair of positions) that permits them to communicate. Furthermore, we require the paths to be optimal according to one of two distance measures: we seek to minimize either the sum of the path lengths or the maximum of the path lengths.

In addition to its application to robot motion planning, our work is applicable to the design of communication networks. Consider the problem of establishing a microwave link between two sites that are not visible to each other. Given that we wish to minimize the total cost of routing cables from the sites to the bases of the towers and that such costs are proportional to the total length of cable needed, our methods can be used to find optimal locations for the two towers (the towers must be visible to each other for a microwave link to be established).

The methods described here complement some of our earlier work. In [4] we discuss how to find optimal paths for two robots that must maintain line-of-sight communication while moving. In that paper we start with robots that are able to communicate and ensure that communication is not broken during the motion. In this paper, we start with robots that are not able to communicate and move them to establish communication. Together, these results allow us to find paths that bring the robots quickly to positions in which they can communicate before continuing on to their destinations, maintaining communication.

In the next section , we give an algorithm that finds optimal rendez-vous paths according to the "min-sum" criterion. In Section 3 we discuss the "min-max" version of the problem.

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2 The Min-Sum Problem

In this section we define the Min-Sum version of the problem more formally and show how to solve it.

We assume that the robots are modeled by points in the plane and that the obstacles to visibility are disjoint simple polygons. We define *freespace* to be the set of all points that are not in the interior of any obstacle. A ***visible rendez-vous*** is an ordered pair of points that can see one another in the sense that the line segment connecting them lies entirely in freespace. A ***line-of-sight*** through a point p in direction θ is a line segment obtained by extending rays from p in directions θ and $-\theta$ until each cannot be extended further without intersecting the interior of an obstacle. We define a ***path*** to be a connected one-dimensional subset of freespace. Finally, we denote the length of a path π by $\mu(\pi)$.

We wish to solve the following problem.

The Min-Sum Visible Rendez-Vous Problem

Instance: An initial configuration (s_1, s_2), and a set S of disjoint simple polygons with a total of n vertices.

Requirement: Find a pair of paths (π_1, π_2) from (s_1, s_2) to a visible rendez-vous (t_1, t_2) such that the total length of the paths $\mu(\pi_1) + \mu(\pi_2)$ is minimized.

It is clear that, in order for a visible rendez-vous to be optimal, it must correspond to a line-of-sight through an obstacle vertex v as shown in Figure 1; otherwise, at least one of the two paths could be shortened (we are assuming that the robots are not visible to one another in the initial configuration). Furthermore, given an orientation of a line-of-sight l through v, the best paths from (s_1, s_2) to a visible rendez-vous on l consist of shortest paths from s_1 and from s_2 to l.

The following lemma shows that not every orientation of a line-of-sight l through v is a candidate line-of-sight for the optimal rendez-vous.

Lemma 2.1 *If an optimal rendez-vous (t_1, t_2) corresponds to a line-of-sight l, then l contains an edge of the visibility graph.*

Proof. Suppose that l is a line-of-sight that passes through only one obstacle vertex v. Then l is free to "rotate" about v (the length of l is adjusted during the rotation so that it extends as far as possible at each end without intersecting the interior of an obstacle). We show that there exists a direction of rotation that improves the value of the objective function, i.e., it decreases the optimal combined distance from s_1 and from s_2 to a visible rendez-vous on l.

Suppose that we have an optimal visible rendez-vous (t_1, t_2) on l and an optimal pair of paths π_1 from s_1 to t_1 and π_2 from s_2 to t_2. We examine the case in which t_1 and t_2 are in the relative interior of l and the distance d_1 from t_1 to v is greater than the distance d_2 from t_2 to v.

The following discussion is illustrated in Figure 2. Let x_1 be the last "turning point" on π_1 prior to reaching t_1, and let x_2 be the corresponding turning point of π_2. For π_1 and π_2 to be optimal, segments $\overline{x_1t_1}$ and $\overline{x_2t_2}$ must be perpendicular to l. Let a_1 and a_2 denote the lengths of these segments. If we rotate l through a small angle in the direction that reduces the distance from x_1 to l, then we get a new visible rendez-vous (t'_1, t'_2) and new distances a'_1 and a'_2. From the figure, it is clear that $a'_1 + a'_2 < (a_1 - b_1) + (a_2 + b_2)$. We also have that $b_1 > b_2$ since $d_1 > d_2$. Thus, $a'_1 + a'_2 < a_1 + a_2$. Since the topology of the optimal paths will not change in some neighborhood of the current orientation of l, small rotations in the direction shown will improve the value of the objective function.

A similar analysis shows that there exists a direction of rotation that will improve the value of the objective function if $d_1 < d_2$ or $d_1 = d_2$. The same result holds if one or both of t_1 and t_2 is an endpoint of l instead of a point in the relative interior of l.

Therefore, l cannot be optimal if it is not "pinned" by two or more vertices. Thus, the optimal line-of-sight must lie along an edge of the visibility graph and must contain its two endpoints. □

The lemma suggests the following algorithm.

The Min-Sum Visible Rendez-Vous Algorithm

1. Construct a shortest path map using s_i as the source for $i \in \{1, 2\}$.
2. Construct the visibility graph $G = (V, E)$ of the set of polygons S.
3. For each extension edge $e \in E$,
    a. Find the line-of-sight e' obtained by extending e in freespace as far as possible in both directions.
    b. Find the length of the shortest path π_1 from s_1 to e' and the length of the shortest path π_2 from s_2 to e' using the shortest path maps.
    c. Compare $\mu(\pi_1) + \mu(\pi_2)$ to the total length of the best pair of paths found so far and update if necessary.

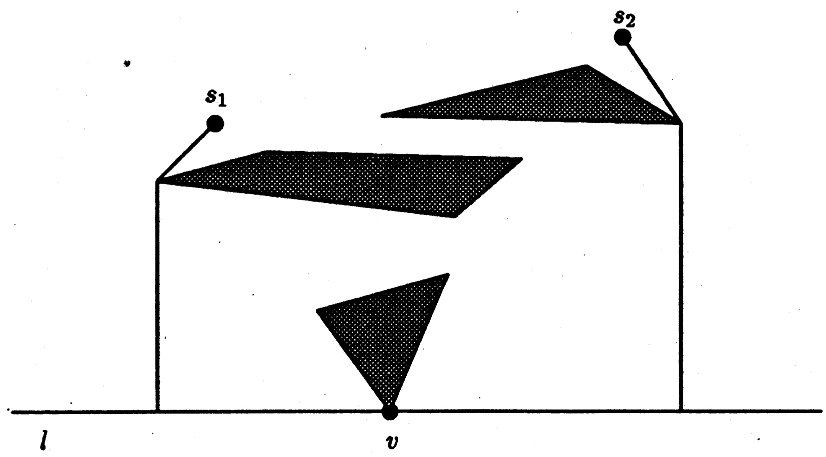

Figure 1: A visible rendez-vous corresponds to a line-of-sight through an obstacle vertex.

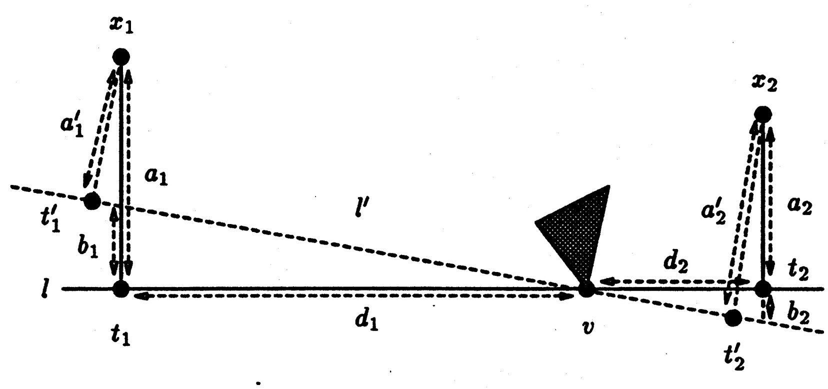

Figure 2: Rotating l in the direction that reduces the distance from x_1 to l, reduces the total length of the optimal paths to a visible rendez-vous on l.

4. Return the best pair of paths found.

The correctness of this algorithm follows from Lemma 2.1. The time and space complexity of the algorithm is analyzed below.

Step 1 takes $O(n^2)$ time and $O(n)$ space using the algorithm of Reif and Storer [7].

Step 2 takes $O(E + n \log n)$ time and $O(E)$ space using the algorithm of Ghosh and Mount [1].

In Step 3 we first find, for each edge e of the visibility graph, the endpoints of the line-of-sight e' that we will refer to as the *extensionedge* of e. This takes constant time, using the representaion of the visibility graph obtained in Step 2. Then we find the lengths of the shortest paths from s_1 and from s_2 to e'. This takes $O(n)$ time as follows.

Given an extension edge e', we locate one endpoint p of e' in the shortest path map and calculate its geodesic distance from s_1. This takes $O(\log n)$ time, using standard point location techniques [6]. We save p as our initial guess of the endpoint of the shortest path from s_1 to e'. Since p is located on the boundary of an obstacle, it lies on the boundary of a cell c in the shortest path map.

We continue by traversing the boundary of c until we reach another point r where e' intersects the boundary of c. We calculate the geodesic distance of r from s_1 the same way we did for p, and compare this distance to the one obtained for p. If r is closer to s_1, then we have a new candidate for the endpoint of the shortest path from s_1 to e'.

Before proceeding, we consider the open line segment between p and r. Since this portion of e' lies completely in c, the shortest path from s_1 to e' can end at a point q on this segment only if the shortest path from the root of c to e' is a segment perpendicular to e' that intersects e' at q. If such a segment exists, the geodesic distance of its endpoint q from s_1 can be calculated in constant time and compared with that of the current candidate for the endpoint of the shortest path from s_1 to e'. If the comparison is favorable to q, it becomes the new candidate endpoint.

We continue to traverse adjacent cells that are pierced by e' and update our representation of the best candidate path from s_1 to e'. When we reach the other endpoint of e', we will have determined the length of the shortest path from s_1 to e'.

After carrying out this process for both s_1 and s_2, we will have determined the sum of the lengths of the shortest pair of paths π_1 and π_2 that permit a visible rendez-vous on e'. The total time needed to traverse the boundaries of the cells pierced by e' is linear in the number of obstacle vertices since the shortest path map is a planar subdivision of freespace. Therefore, Step 3, iterating over all edges of the visibility graph, takes $O(En)$ time.

In summary, we have the following theorem.

Theorem 2.1 *The Min-Sum Visible Rendez-Vous Problem can be solved in $O(En)$ time and $O(E)$ space.*

3 The Min-Max Problem

Now we turn to the Min-Max version of the problem.

The Min-Max Visible Rendez-Vous Problem

Instance: An initial configuration (s_1, s_2), and a set S of disjoint simple polygons with a total of n vertices.

Requirement: Find a pair of paths (π_1, π_2) from (s_1, s_2) to a visible rendez-vous (t_1, t_2) such that the maximum length of the paths $\max\{\mu(\pi_1), \mu(\pi_2)\}$ is minimized.

In the Min-Max problem, Lemma 2.1 no longer holds. In Figure 3, each of the indicated paths has the same length and together they form an optimal min-max pair. The line-of-site l containing the optimal visible rendez-vous (t_1, t_2), however, is not "pinned" by two vertices. Thus we must now consider paths to a line-of-sight l that passes through only a single vertex and calculate the optimal orientation of such a line.

Given a line l (an infinite line, not a line-of-sight) that passes through a vertex v, we find the optimal orientation of l by solving $O(n^2)$ subproblems. In each subproblem, we choose vertices x_1 and x_2 and assume that they are the last turning points on the shortest paths to l from s_1 and s_2, respectively. With this assumption, the length of the shortest path from s_i to l is the sum of the geodesic distance from s_i to x_i and the Euclidean distance from x_i to l. Only the second term varies with the orientation $l(\theta)$ of l.

It is easy but tedious to solve for the optimal orientation θ^* of l using elementary algebra. Several cases arise because the shortest path from x_i to l^* may intersect l^* at a point in the interior of freespace or it may be determined by the intersection of l^* and an obstacle edge. The number of possible cases is bounded by a fixed constant, however, so the solution can be found in constant time.

We ignore the obstacles when solving for the optimal orientation of l. Thus we must check whether the solution we find is feasible. Let $l^* = l(\theta^*)$ and let (t_1, t_2) be the optimal visible rendez-vous on l^* using x_1 and x_2 as final turning points. For feasibility, we must ensure that each of the pairs of points (t_1, t_2),

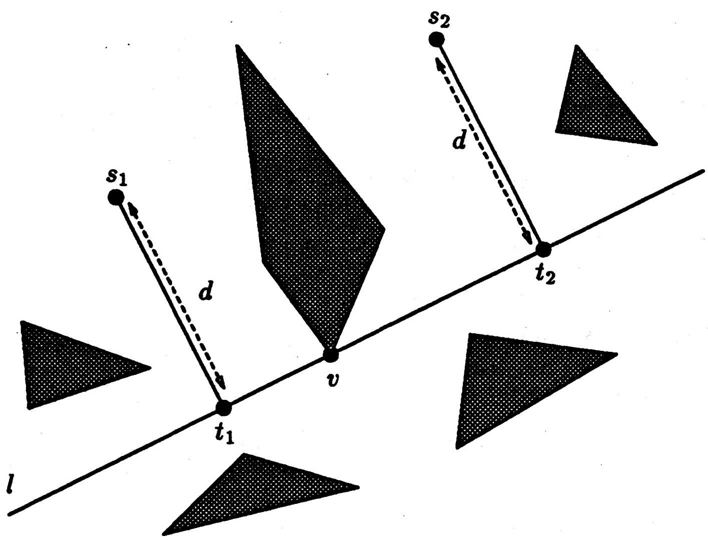

Figure 3: An optimal min-max visible rendez-vous may occur at an orientation of a line-of-sight l that is not "pinned" by two or more vertices.

(x_1, t_1), and (x_2, t_2) corresponds to a visibility segment, i.e., a line segment lying entirely in freespace.

If the first test fails, i.e., if (t_1, t_2) is not a visible rendez-vous, then l^* intersects an obstacle in the interval between t_1 and t_2. In this case, the optimal *feasible* orientation θ of l occurs when $t_1(\theta)$ and $t_2(\theta)$, the closest points of $l(\theta)$ to x_1 and x_2, are aligned with v and another obstacle vertex. In other words, $t_1(\theta)$ and $t_2(\theta)$ lie on an extension of a visibility graph edge. We find visible rendez-vous paths of this form separately using the method described in Section 2.

On the other hand, if t_i is not visible to x_i for some $i \in \{1, 2\}$, then our choice of x_i's was not optimal, i.e., x_i is not the last turning point on a shortest path from s_i to t_i. In either case, if the feasibility test fails, we drop the current pair of x_i's from consideration.

With these discussions as a prelude, we now describe the algorithm for the Min-Max problem.

The Min-Max Visible Rendez-Vous Algorithm

1. Construct a shortest path map using s_i as the source for $i \in \{1, 2\}$.
2. Construct the visibility graph by finding the visibility polygon of each vertex.
3. Find the best visible rendez-vous on an extension of a visibility graph edge as in Step 3 of the Min-Sum Algorithm using the Min-Max optimization criterion.
4. For each ordered triple (x_1, v, x_2),
   a. Find the Min-Max value of the best pair of paths (π_1, π_2) from s_1 and s_2 to a line l through v such that x_1 and x_2 are the last turning points on π_1 and π_2.
   b. Check the feasibility of the visible rendez-vous (t_1, t_2) determined by (π_1, π_2), using the visibility polygons of x_1, x_2, and v.
   c. If the current triple is feasible, compare $\max\{\mu(\pi_1), \mu(\pi_2)\}$ to the best value found so far, and update if necessary.
5. Return the best pair of paths found.

This algorithm leads to the following theorem.

Theorem 3.1 *The Min-Max Visible Rendez-Vous Problem can be solved in $O(n^3 \log n)$ time and $O(n^2)$ space.*

Proof. The algorithm given above solves the problem correctly as discussed above and can be implemented to run in $O(n^3 \log n)$ time using $O(n^2)$ space as follows. Computation of the visibility polygons and the visibility graph in Step 2 takes $O(n^2)$ time and $O(n^2)$ space using the algorithm of Welzl [8]. Steps 1 and 3 take $O(En)$ time just as they did in the Min-Sum case. In Step 4, we solve $O(n^3)$ subproblems. In each subproblem, the cost of performing the feasibility check dominates the cost of other operations and can be done in $O(\log n)$ time using binary search on the visibility polygons about x_1, x_2 and v. □

4 Extensions and Open Problems

We have shown that the Min-Sum and Min-Max versions of the Rendez-Vous problem can be solved efficiently. Now we give several extensions and open problems.

One extension to the problems discussed here is that in which we allow "Steiner Points" in the visibility link between two agents. In other words, we could require that the agents move to a configuration in which they can be connected by a path of link distance no greater than k for some fixed k. This version of the problem models the situation in which we want to connect two stations by microwave towers and several towers in succession may be used to relay messages between stations. The Min-Sum version of this problem, in which we minimize the sum of the cable lengths needed to connect the stations to the first and last towers, can be solved by combining our techniques with known methods for determining the link distance between two edges of the visibility graph [3].

Another extension of our problem is that in which we require that the agents not only see each other but also be within distance D of each other in order to communicate. It is not yet clear whether this extra requirement increases the complexity of the problem.

If we require that agents need to be within distance D to communicate, without requiring that they be visible to one another, the problem becomes one of finding the best placement of a circle of radius D so that there exist optimal obstacle-avoiding paths from the initial positions of the agents to two points on the circle. In this case we call the desired pair of final positions an *optimal proximity rendez-vous.*

When $D = 0$ the problem is trivial since any point on the shortest path between the initial positions is optimal for the min-sum version of the problem, and the "midpoint" of the shortest path is optimal for the min-max version of the problem. We call the optimal meeting place in this version an *optimal physical rendez-vous.*

Finding an optimal physical rendez-vous becomes more interesting if the number of agents is greater than two and we wish to minimize the maximum geodesic distance from an initial position to the point where the physical rendez-vous takes place. In this case, we have a generalization of the well-known 1-center problem. In the Euclidean plane this problem is called the smallest-enclosing-circle problem [6] since the solution is the center of a circle of minimum radius that contains all of the intial locations. The solution in the plane can be found in linear time using a variation of Megiddo's linear programming algorithm [2]. In a simple polygon, the 1-center or "geodesic center" can be found in $O(n \log n)$ time [5]. The question of whether there exists an efficient algorithm for finding the geodesic center of a set of points among polygonal obstacles, however, seems to be still open.

Extending our visible rendez-vous results to the k-agent case for $k \geq 3$ seems to be challenging, as well. Even in the 3-agent min-sum version, the local optimality criterion that the line-of-sight connecting two agents must be an extension of a visibility graph edge no longer holds. It is not yet clear whether the problem is discrete or continuous.

References

[1] S. K. Ghosh and D. M. Mount. An output sensitive algorithm for computing visibility graphs. In *Proceedings of the 28th Annual IEEE Symposium on the Foundations of Computer Science*, pages 11–19, 1987. To appear: *SIAM Journal on Computing.*

[2] N. Megiddo. Linear programming in linear time when the dimension is fixed. *Journal of the ACM*, 31:114–127, 1984.

[3] J. S. B. Mitchell, G. Rote, and G. Woeginger. Minimum-link paths among obstacles in the plane. In *Proceedings of the 6th Annual ACM Symposium on Computational Geometry*, pages 63–72, 1990.

[4] J. S. B. Mitchell and E. L. Wynters. Optimal motion of covisible points among obstacles in the plane. In *Proceedings of the 2nd Canadian Conference on Computational Geometry*, pages 116–119, 1990.

[5] R. Pollack, M. Sharir, and G. Rote. Computing the geodesic center of a simple polygon. *Discrete and Computational Geometry*, 4:611–626, 1989.

[6] F. P. Preparata and M. I. Shamos. *Computational Geometry - an Introduction.* Springer Verlag, New York, 1985.

[7] J. H. Reif and J. A. Storer. Shortest paths in euclidean spaces with polyhedral obstacles. Technical Report CS-85-121, Brandeis University, Waltham, Mass., 1985.

[8] E. Welzl. Constructing the visibility graph for n line segments in $O(n^2)$ time. *Information Processing Letters*, 20:167–171, 1985.