Maximal Outerplanar Graphs are Relative Neighbourhood Graphs

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1 Introduction

Relative neighbourhood graphs (RNGs) were first mentioned by Toussaint [11] as a way of describing the structure of a set of points for the purpose of pattern recognition. The RNG on a finite set of points fixed in the plane is a graph where two points are adjacent vertices in the RNG if they are closer to each other than they are to any other point. A formal definition is the following:

Definition 1 *The RNG on a vertex set V of fixed points in the plane has an edge set E where* $(p,q) \in E$ *if, for all* $z \in V, z \neq p, q$,

$$d(p,q) \leq \max\{d(p,z), d(q,z)\}$$

where $d(p,q)$ *is the Euclidean norm.*

A lot of work has been done on finding fast algorithms for computing the RNG but little work has been done on characterizing them. A graph can be realized as a RNG if it is possible to place points in the plane so that the RNG on these points gives the original graph. Urquhart [12] has investigated some of the properties of the RNG and proves that n-cycles and wheel graphs with seven or more vertices and any tree of degree three can be realized as RNGs but wheel graphs with six or fewer vertices can not. We prove that all maximal outerplanar graphs can be realized as RNGs.

RNGs are part of a spectrum of proximity graphs which include minimum spanning trees, Gabriel graphs, sphere-of-influence graphs, Delaunay triangulations, lune- and sphere-based β-skeletons, and γ-neighbourhood graphs. Veltkamp [13] gives an overview of these proximity graphs and describes the γ-neighbourhood graph. Further work on properties of proximity graphs has been done by Cimikowski [3]. Supowit [10] looked at faster methods for computing the RNG on a set of points and applied it to minimum spanning trees. Agarwal and Matoušek [1] have extended the research into three dimensions. A survey of results on neighbourhood graphs has been published by Jaromczyk and Toussaint [6]. Dillencourt [4] has shown that maximal outerplanar graphs can be realized as Delaunay triangulations. Bose, Lenhart, and Liotta [2] are presently doing work on characterizing proximity trees.

Section 2 contains an alternative definition of the RNG and some preliminary lemmas and examples; Section 3 contains the proof that maximal outerplanar graphs can be realized as RNGs.

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2 Definitions and Lemmas

Definition 1 says that two vertices are adjacent in the RNG of a set of points if no point is closer to the pair than they are to each other. This indicates that there is an area dependent on the two points which must be empty if they are to be connected. Thus, another way of describing the RNG is:

Definition 2 *The points p and q are adjacent vertices in the RNG defined on a vertex set V of fixed points in the plane if the* lune(p, q) *is empty, where*

$$\text{lune}(p,q) = \{z \in \mathbf{R}^2 : d(p,z) < d(p,q) \text{ and } d(q,z) < d(p,q)\}$$

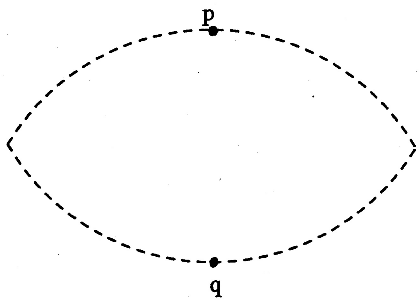

Figure 1: Lune (p, q) for RNG

Note that points may occur on the boundary of the lune and the two points will still be adjacent. This way three equidistant points will be pairwise adjacent. It has been shown by Toussaint [11] that any graph realizable as an RNG is planar. The simplest planar graphs are triangles so we look at their representation.

Definition 3 *An isosceles triangle where the edges of equal length are no shorter than the third edge is called a* pointy *triangle.*

Lemma 1 *(Urquhart [10, Lemma 2.2]) The RNG on three points in the plane gives a triangle if and only if these points form a pointy triangle.*

Therefore, if a graph is made up of triangles, these triangles would all have to be pointy triangles in the RNG representation. This is not sufficient, however. Take, for example, a vertex a with neighbours $a_1, \ldots, a_6$ where (a_i, a_{i+1}) is an edge for $1 \le i \le 5$. Place $a_1, \ldots, a_6$ on a circle around a such that the angle $\angle a_i a a_{i+1} = 60°$. Thus triangle $\triangle a_i a a_{i+1}$ is a pointy triangle but $\triangle a_1 a a_6$ is also a pointy triangle, so the RNG includes the edge (a_1, a_6).

A graph is maximal planar if it is planar but adding any edge makes it non-planar. Thus maximal planar graphs are graphs whose faces are all triangles. Not all maximal planar graphs are realizable as RNGs because the wheel graph with four vertices, or K_4, is not.

Definition 4 *A graph is outerplanar if it can be embedded in the plane in such a way that all of its vertices are on the same face. A graph is maximal outerplanar if it is outerplanar but adding any edge makes it not outerplanar.*

A maximal outerplanar graph that has three or more vertices can be embedded so that the face which contains all the vertices is the outer face, the edges of the graph are the chords of the outer facial cycle, and all faces except the outer face are triangles.

The next section contains our proof that all maximal outerplanar graphs are RNGs.

3 Maximal Outerplanar Graphs

Theorem 1 *Maximal outerplanar graphs can be realized as RNGs.*

PROOF: Look at an embedding of the graph where the face which contains all the vertices is the outer face. We will preserve the topology of this embedding. The proof will be by induction on the number of vertices of the given graph. To this end we will strengthen the statement of the theorem:

Theorem 2 *Given an embedding of a maximal outerplanar graph G with all the vertices on the outer face, a fixed edge $e = (u, u_1)$ on the outer face with u appearing before u_1 in a [counter]clockwise ordering of the outer face, and a right-angled triangle $T = \triangle rst$ (ordered [counter]clockwise and with right angle at s), we can represent G as a RNG so that all the points lie inside or on triangle T, and u and u_1 are identified with r and s respectively.*

PROOF: We will consider the case of a counterclockwise ordering. Assume G has at least 3 vertices since the graph with two connected vertices can obviously be realized as a RNG. Let the angle at r be α. We may assume $\alpha < 60°$ otherwise we may simply embed the graph in a smaller triangle contained in the original. Let the neighbours of u be $u_1, \ldots, u_k$ in a counterclockwise ordering where u_i is adjacent to u_{i+1}, $i = 1, \ldots, k-1$. (For the rest of the proof assume that $i = 1, \ldots, k-1$ where k is the number of neighbours of u.) Identify u with r and u_1 with s. Create a fan from edge (u, u_1) with u at its center so that the triangles $\triangle uu_iu_{i+1}$ are equally sized pointy triangles. Let the angle that these pointy triangles make at vertex u be $\alpha_1 = \alpha/(k-1)$. This fan is contained in the right-angled triangle $\triangle uu_1t$. All the neighbours $u_1, \ldots, u_k$ of u are on an arc centered at u with radius $d(u, u_1)$. This will be called the *arc of u*. See Figure 2.

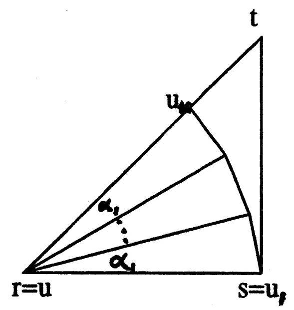

Figure 2: Fan of u

Now u and its k neighbours have been placed and no other vertex of the graph G is adjacent to u. Thus the rest of G must attach to edges (u_i, u_{i+1}). Let G_i be the part of the graph attached to and including edge (u_i, u_{i+1}). Each G_i is maximal outerplanar and inherits an embedding from G. Note that (u_i, u_{i+1}) is an edge of the outer face of G_i and u_i appears before u_{i+1} in a clockwise ordering of the outer face. Thus G consists of the subgraphs $G_1, \ldots, G_{k-1}$, the vertex u, and the edges from u to $u_1, \ldots, u_k$.

Let (u_i, u_{i+1}, t_i) form a clockwise right-angled triangle T_i with the right angle at u_{i+1} and an angle of $\alpha_1/2$ at u_i. By the induction hypothesis, G_i can be realized as a RNG inside triangle T_i, with the edge (u_i, u_{i+1}) as fixed.

This completes our construction of the representation of maximal outerplanar graphs. In the remainder of the proof we will show that this construction works, first proving that the points we constructed are contained in $T = \triangle rst$ and then proving that the RNG of the points is G.

By construction, u and its neighbours are contained in T. It suffices to verify that the extreme vertices of the T_i's, t_1 and t_{k-1}, are contained in T. The angle $\angle t_1 u_1 u = \alpha_1/2 + (90° - \alpha_1/2) = 90°$, so t_1 is on edge (u, t) of T. The angle $\angle u u_k t_{k-1} = 90° + (90° - \alpha_1/2) < 180°$, so t_{k-1} is inside T. See Figure 3.

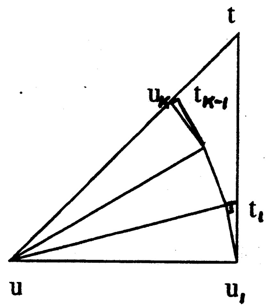

Figure 3: Triangles T_1 and T_k

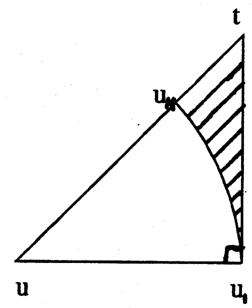

Figure 4: Far area of T

We will now prove that the construction gives the correct RNG, beginning with a preliminary claim. Call the area enclosed by the arc of u and the edges (u_1, t) and (t, u_k) the *far area* of T. See Figure 4.

Claim 1 *The construction puts all vertices of G except u in the far area of T and the neighbours of u are the only points on the arc of u, in particular, if x is not a neighbour u_i then $d(x, u) > d(u_i, u)$.*

PROOF: It suffices by induction to observe that the far area of T_i is contained in the far area of T. □

We will now show that the RNG of the constructed points is in fact G.

1) All edges of G are in the RNG of the constructed points: Since an edge of G is either an edge (u, u_i), or lies in some subgraph G_i, it suffices by induction to prove that
a) (u, u_i) is an edge in the RNG
b) points of $G \setminus G_i$ do not affect the RNG of points of G_i.

Proof of a): All neighbours of u are the same distance away from u so they can at most be on the lune of u and another neighbour. By Claim 1, all other points of G must be inside the far area of T and are farther away from u than u_i. Thus they are outside the lune(u, u_i).

Proof of b): It suffices to prove that the lune of two adjacent points x and y of G_i does not contain u nor any points of G_{i+1} or G_{i-1}. Let y be in the far area of T_i and let x be any other point in G_i. We begin by proving that

$$d(x, y) \leq d(u_i, u_{i+1}):$$

If $x = u_i$ then y is on the arc of $x = u_i$ and $d(x, y) = d(u_i, u_{i+1})$. If $x \neq u_i$ then x and y are both in the far area of T_i. Let v_i be the last vertex on the arc of u_i, $v_i \neq u_{i+1}$. Now $\triangle u_{i+1}u_iv_i$ is isosceles so $\angle u_{i+1}v_iu_i < 90°$, making $\angle u_{i+1}v_it_i > 90°$. Since $\triangle u_{i+1}v_it_i$ is obtuse, the longest edge of $\triangle u_{i+1}v_it_i$ is the edge (u_{i+1}, t_i). Any two points, namely x and y, in the far area of T_i, are contained in $\triangle u_{i+1}v_it_i$ so $d(x, y) \leq d(u_{i+1}, t_i)$. Now $d(u_{i+1}, t_i) < d(u_i, u_{i+1})$ since they are two sides of a right-angled triangle and $\angle\alpha_1/2 \leq 30°$. Thus $d(x, y) \leq d(u_i, u_{i+1})$. See Figure 5.

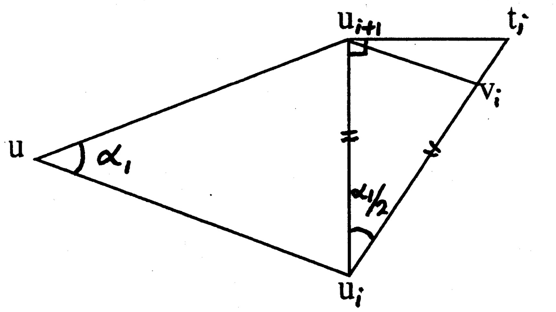

Figure 5: Triangle T_i

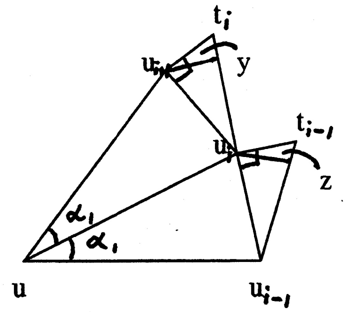

Figure 6: Angle yu_iz

Consider u. Since y is in the far area of T_i, by Claim 1, $d(u, y) > d(u, u_i)$, and since $\angle u_{i+1}uu_i \leq 60°$, $d(u, u_i) \geq d(u_i, u_{i+1})$. As well, $d(u_i, u_{i+1}) \geq d(x, y)$ as proven above, so $d(u, y) > d(x, y)$ showing that u does not affect the adjacency of any two points in G_i.

Consider a point z of G_{i-1} or G_{i+1} as placed in T_{i-1} or T_{i+1} respectively. We can assume that z is in the far area of T_{i-1} or T_{i+1}. We will prove $d(z, y) \geq d(u_i, u_{i+1})$. Since this is symmetric with respect to z and y, we can assume that $z \in G_{i-1}$ and $y \in G_i$. See Figure 6. Look at $\angle yu_iz$:

$$\begin{aligned}\angle yu_iz &= 360° - \angle t_{i-1}u_iu_{i-1} - \angle u_{i-1}u_iu - \angle uu_iu_{i+1} - \angle u_{i+1}u_it_i + \angle yu_it_i + \angle t_{i-1}u_iz \\ &\geq 360° - 90° - (90° - \alpha_1/2) - (90° - \alpha_1/2) - \alpha_1/2 \ \geq \ 90° + \alpha_1/2 \\ &> 90°.\end{aligned}$$

Thus $\triangle yu_iz$ has an obtuse angle at $\angle yu_iz$, making $d(y, z) > d(y, u_i)$ and since y is in the far area of T_i, $d(y, u_i) \geq d(u_i, u_{i+1})$ giving $d(y, z) > d(u_i, u_{i+1})$ for all $y \in G_i$ and $z \in G_{i-1}$ or $z \in G_{i+1}$.

As proved above, $d(x, y) \leq d(u_i, u_{i+1})$. Thus, for any $z \in G_{i-1} \cup G_{i+1}$, $d(y, z) > d(x, y)$, and z is outside the lune(x, y).

2) No other edges are in the RNG: Let x and y be non-adjacent vertices of G. We will show that x and y are not adjacent in the RNG.

If $x = u$ then by Claim 1, y must be in the far area of T and since $y \neq u_1, \ldots u_k$, we have $d(x, y) > d(x, u_i)$. If x and y are both in G_i, then by induction, the RNG of G_i has no edge joining x and y. Otherwise $x \in G_i$, $y \in G_j$, $i \neq j$. It suffices to consider $y \in G_{i+1} \setminus G_i$ and $x \neq u_{i+1}$. Then y is in the far area of T_{i+1} and $\angle yu_{i+1}x > 90°$, as proved above. So $d(y, u_{i+1}) < d(x, y)$ and $d(x, u_{i+1}) < d(x, y)$, which means that u_{i+1} is in the lune(x, y). Thus x and y are not joined by an edge in the RNG.

Therefore, since we have given a construction for the vertices of a maximal outerplanar graph G such that the RNG on those points is exactly G, all maximal outerplanar graphs are realizable as RNGs. □

4 Discussion and Further Results

Maximal outerplanar graphs have been shown to be representable in several geometric ways: El-Gindy proved that any maximal outerplanar graph is a visibility graph (see [8] for an exposition); Dillencourt [4] showed that maximal outerplanar graphs can be realized as Delaunay triangulations. We have just shown that maximal outerplanar graphs can be realized as RNGs. This result is most closely related to Dillencourt's because the RNG of a given set of points is always contained in the Delaunay triangulation of that set of points. Thus is it natural to ask whether a single construction could prove both results. However, Dillencourt [5] has shown that a maximal outerplanar graph with four or more ears cannot be realized simultaneously as a RNG and as a Delaunay triangulation on the same set of fixed points.

Using the same construction, we can also show that maximal outerplanar graphs are Gabriel graphs [9]. (See Matula and Sokal [7] for the definition of the Gabriel graph.)

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