Dominance Drawings of Bipartite Graphs*

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Abstract

Let $G = (S, T, E)$ denote a directed bipartite graph with S a set of sources and T a set of sinks. Using vectors $\mathbf{s} = (s_1, s_2, ...s_k)$ and $\mathbf{t} = (t_1, t_2, ...t_k)$ in R^k to represent the elements $s \in S$ and $t \in T$ we can geometrically characterize the properties of G. We consider the case where $s_i \leq t_i$, $i = 1, 2, ..., k$, (with strict inequality in at least one coordinate) if and only if the directed edge $(s, t) \in E$. Such an assignment of coordinates to vertices is known as a *dominance drawing* of G, because all edges in the graph are geometrically characterized by the dominance relation. We give upper and lower bounds for the dimension requirements of a dominance drawing for bipartite graphs. We also present an asymptotically optimal algorithm to obtain a 2-dimensional dominance drawing of a bipartite graph whenever such a drawing exists.

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1 Introduction

Geometric representations can be valuable tools for characterizing properties of graphs. A good drawing of a graph can be used to display many different properties in many different ways. See [5] for a comprehensive survey of methods that can be used to obtain good graph drawings. We will consider graphs that represent partial orders and discuss a particular method of drawing the graphs to geometrically display the transitive closure property of the partial order.

We begin with some definitions. Let $G = (V, E)$ denote a directed graph with vertex set V and edge set $E = \{(x, y) : x, y \in V\}$. We say that the edge (x, y) is *directed* from x to y. A vertex y is *reachable* from x if there is a *path* of one or more directed edges that lead from x to y. An unreachable vertex is a *source* and an unreaching vertex a *sink*. A *directed acyclic digraph* (DAG) is a directed graph with no vertex reachable from itself. A directed edge (x, y) is *transitive* if y is also reachable from x by a path not using (x, y). A *transitive digraph* is a DAG that includes all transitive edges. A *straight line planar drawing* is a two dimensional representation of a graph where vertices are represented by points in a plane and edges are represented by straight line segments that do not cross. A graph is *planar* if and only if it admits a planar straight line drawing.

A partial order P of a finite set X is a transitive and non-reflexive binary relation on X. A partial order can be represented by a *transitive digraph*, $G(P)$ on the elements of X, The *dimension* $d(P)$ of a partial order P [4] is the minimum number of linear orders whose intersection is P. There is a direct interpretation of $d(P)$ as it pertains to its associated graph. We can use vectors $\mathbf{x} = (x_1, x_2, ...x_k)$ to represent each vertex x of $G(P)$, so that $x_i \leq y_i, i = 1, 2, ..., k$, (with strict inequality in at least one coordinate) if and only if y is reachable from x in $G(P)$. In general we call such an assignment of coordinates to vertices a *dominance drawing*, because all edges in the graph (and transitive closure) are geometrically characterized by the dominance relation. There is a k-dimensional dominance drawing of $G(P)$ if and only if $d(P) = k$. Thus the notions of the partial order dimension and the dominance drawing dimension are equivalent, so we will also use the notation $d(G)$ to denote the dominance drawing dimension of the graph G.

Hiraguchi [7] showed that $d(P) \leq |X|/2$, whilst Yannakakis [12] proved that it is NP-complete to decide whether the dimension of a partial order is at most k, for $k \geq 3$.

Necessary and sufficient conditions for $d(P) = 2$ are given by [4]. Let $\overline{G}$ denote the complement of an undirected version of G, that is, (x, y) is an edge of $\overline{G}$ if neither (x, y) nor (y, x) are edges of G. Dushnik and Miller [4] proved that $d(P) = 2$ if and only if there is a way to orient the edges of $\overline{G}$ to make it a transitive digraph. Graphs that admit such an orientation of their edges are called *transitively orientable*. A polynomial time algorithm to recognize transitively orientable graphs is given in [10]. However, the complexity analysis of the algorithm is not explicitly discussed. Nevertheless, by combining the characterization in [4] and the algorithm in [10] a polynomial time algorithm is available to obtain a two dimensional dominance drawing of a transitive digraph if such a drawing is possible.

Kameda [8] defines a class of graph that always admits a two dimensional dominance drawing and gives an algorithm that is proportional to the size of the input graph to obtain the drawing. Kameda shows that planar DAGs with all sources and sinks embedded on the same external face so that the sources and sinks divide the boundary of this face into two contiguous parts always admit a two dimensional dominance drawing. In [11] Tamassia observes that Kameda's algorithm can be applied to the so called *reduced planar st-graphs.* In [11] straight line planar dominance drawings are obtained from DAGs with no transitive edges and with a single source and a single sink.

Another class of graph that admits a two dimensional dominance drawing is the class of *series-parallel digraphs* [1]. A series parallel digraph is recursively defined as, a single edge joining two vertices, and if G_1 and G_2 are two series parallel digraphs then they can be composed in series by unifying the source of G_2 with the sink of G_1 or composed in parallel by unifying the source of G_1 with the source of G_2 and unifying the sink of G_1 with the sink of G_2. In [1] it is shown that all series parallel digraphs admit a planar two dimensional drawing.

In the full paper, we present results pertaining to transitive directed bipartite graphs. Let $G = (S, T, E)$ denote a directed bipartite graph with S a set of *sources* and T a set of *sinks*, so that all edges (s, t) are directed from a source to a sink. The transitive property of G is trivially realized since the maximum path length in G is one. Let $n = min(|S|, |T|)$. We show that $d(G) \leq n$, and that there exist bipartite graphs G where $d(G) = n$. In [12] it is proved that deciding whether the dimension of a dominance drawing of a transitive bipartite graph is at most 4 is NP-complete. We show that a two dimensional dominance drawing for a transitive bipartite graph can be obtained in polynomial time. The complexity of our algorithm is linear in the size of the input and is thus optimal to within a constant of proportionality. This compares favourably with the complexity of the algorithm in [6] [10]. At the moment the case for three dimensional dominance drawings of transitive bipartite graphs remains open.

2 Sketch of the Algorithm

Let $G = (S, T, E)$ be a transitive bipartite graph with $S = (s_1, s_2, \ldots, s_{|S|})$ a set of sources, and $T = (t_1, t_2, \ldots, t_{|T|})$ a set of sinks. We use $N(s) = \{t : (s_i, t_j) \in E\}$ to denote the *neighbourhood* of s. Let $\mathcal{N} = \{N(s) : s \in S\}$, and let $\mathcal{M} = \{N(s) - N(w) : s, w \in S, N(w) \subset N(s)\}$, then $\mathcal{I}(S, T) = \mathcal{N} \cup \mathcal{M}$. Let $\Pi(\mathcal{I}(S, T))$ denote the collection of all permutations of T, π, such that members of each subset $I \in \mathcal{I}(S, T)$ are contiguous in π. Our algorithm to obtain a two dimensional dominance drawing of the graph G is based on the following characterization of two dimensional transitive bipartite graphs.

Theorem 1 *The dimension of $G = (S, T, E)$ is less than or equal to two if and only if $\Pi(\mathcal{I}(S,T))$ is not empty.*

Consider a set X and a set of subsets of X, Ξ. Booth and Leuker [2] present the so called PQ-tree algorithms that can be used to determine the family of permutations of X so that every subset $\xi \in \Xi$ is contiguous in the family of permutations. The resulting computational complexity is linear in the size of the input. Thus it appears that an expedient solution to our problem is to compute the set $\mathcal{M}$ and subsequently the set $\mathcal{I}(S,T)$ and apply the PQ-tree algorithm. However, the size of $\mathcal{M}$ may be $O(|S|^2)$. This problem is not insurmountable as we can restrict our attention to a linear sized subset of $\mathcal{M}$. Consider the case where we have a maximal sequence $N(s_1) \subset N(s_2) \subset \cdots \subset N(s_k)$, then we only need to consider the set $N(s_k) - N(s_1)$. However, the task of computing this reduced subset of $\mathcal{M}$ approaches the conceptual difficulty of presenting a new approach without using PQ-trees. We present a new algorithm and skirt the problem of computing a reduced subset of $\mathcal{M}$. The data structure we use to represent the family of permutations $\Pi(\mathcal{I}(S,T))$ is influenced by the PQ-tree, but it is specially tailored for this problem and is much simpler.

We define a *boxlist* of a set T, $\mathcal{B}(T)$, as the empty list, or a linked list consisting of one or more boxes, where each box contains a subset of T, and the boxes form an exact cover of T, that is, the union of the boxes in the boxlist is T and each element of T appears in exactly one box. If we fix intra-box ordering of elements then a rear to front, or front to rear, traversal of $\mathcal{B}(T)$ corresponds to a permutation of T. A permutation π of T is *consistent* with $\mathcal{B}(T)$ if the elements within boxes can be ordered so that a traversal of $\mathcal{B}(T)$ is equal to π. Let $F(\mathcal{B}(T))$ be used to denote the set of all permutations that are consistent with $\mathcal{B}(T)$. Given a graph $G = (S, T, E)$ our algorithm begins with a boxlist representing all permutations of T, that is, the boxlist consists of exactly one box that contains T itself. If the graph has a two dimensional dominance drawing then the algorithm exits with a non-empty boxlist such that $F(\mathcal{B}(T)) = \Pi(\mathcal{I}(S,T))$ otherwise the algorithm exits with $\mathcal{B}(T) = \emptyset$, an empty list.

The principal operation performed on a boxlist is to add *constraints* to $\mathcal{B}(T)$ that are associated with a neighbourhood of a source, $N(s)$. The constraints are substrings, or *intervals* within the permutations of $\Pi(\mathcal{I}(S,T))$. Thus $\mathcal{B}(T)$ is constrained so that the interval associated with $N(s)$ will be contiguous in all permutations of $F(\mathcal{B}(T))$. We show that the adding of constraints can be scheduled so that it is easy to check whether an interval is contiguous within $\mathcal{B}(T)$, and that $N(s) - N(w)$ is contiguous for all w such that $N(w) \subset N(s)$. We maintain an overall linear complexity by avoiding explicit sorting.

Our algorithm development is summarized in the following theorem.

Theorem 2 *Given a connected transitive bipartite graph* $G = (S, T, E)$*, our algorithm returns* $F(\mathcal{B}(T)) = \Pi(\mathcal{I}(S, T))$*. The algorithm can be implemented to run in* $O(|S| + |T| + |E|)$ *time and space, and this is within a constant multiple of optimal.*

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