Topological Characterization of Hwang's Optimal Steiner Trees

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Abstract

We present in this paper a topological characterization of Hwang's optimal Steiner trees. This characterization permits to determine configuration yielding optimal Steiner components, and this only by considering the topology of terminals' position. We present an equation defining exactly the number of Steiner points of an optimal Steiner tree following its number of terminals and so called optimal Steiner components. Finally, we present a simple procedure to compute the set of points that can potentially be Steiner points of an optimal Steiner tree for some set of terminals. One interesting property of this procedure is that it relies only on topological relation of the terminals' position.

1. Introduction

Let P be a set of points in the plane and H be a tree in the plane having V as vertices (points). H is a *spanning tree* of P when $P=V$, and a *Steiner tree* of P when $P \subseteq V$. The length of an edge in H is the distance in the plane between two points (vertices) and the length of H is the total length of all its edges. For a set P, a *minimal spanning tree* (MST) is the spanning tree of minimal length and the *Steiner minimal tree* (SMT) is a Steiner tree of minimal length.

An *Euclidian SMT* (ESMT) and *rectilinear SMT* (RSMT) are SMT problems with edge's length measured respectively in the euclidian distance metric and the rectilinear distance metric. The distance between two points p_1 and p_2 in the rectilinear metric is defined as

$$d(p_1,p_2) = |x_1-x_2| + |y_1-y_2|,$$

where (x_i,y_i) are the Cartesian coordinates of p_i.

The MST and SMT problems are very well known. The MST problem may be solved in $O(n \log n)$ time in the plane where $n=|P|$, and both ESMT and RSMT problems are NP-complete [4,5].

In this paper we present a topological characterization of Hwang's theorem [7] for rectilinear Steiner minimal tree (RSMT). This theorem has been used in several papers [1,3,11]. An extended survey of Steiner tree problems is presented by Hwang [8]. This problem has a direct application in VLSI routing for which interconnected wires are constrained to consist of horizontal or vertical line segments in a plane.

The presented topological characterization of optimal Steiner trees (OST) was undertaken to reduce de number of Steiner points that need to be considered to compute an optimal solution. We show that the number of Steiner points added to the number of so called *optimal Steiner components* is always equal to $n-1$ in an OST, where n is the number of terminals. This permits the design of more efficient algorithms for obtaining exact or approximate solutions of instances of this problem. They have already been exploited by a neural network implementation of an approximation algorithm [2].

We present in the next section notation and fundamental terminology that are used in the rest of the paper. In section 4 we present our characterization based on previous work reported in section 3. Finally, in section 5, we present a simple procedure based on our characterization. It permits to determine a set of necessary Steiner point that can composed an Optimal Steiner component. The main property of this procedure is that it relies only on topological relation of the terminals' position.

2. Notation and terminology

We use the same terminology as in [10] and we refer the interested reader to this paper for more information on this subject. It is summarized in this section. We have as input a set T of points in the plane that are called *terminals*. A point having a degree greater than 2 in a rectilinear Steiner tree is called a Steiner point. The distance between two points is measured in the L_2 metric. A *line* is a sequence of one or more adjacent, collinear segments with no terminal in its relative interior. A *complete line* is a line of maximal length. A *corner* is a degree-two node which is not a terminal. A *complete corner* is a corner with two incident complete lines each terminated by a terminal. The incident lines of a complete corner are their *legs*.

Two transformations are defined for rectilinear Steiner trees: flipping and sliding. Each transformation maps one tree to another without moving the positions of terminals and without increasing the length of the tree. Depending on the direction, there are four slides and four flips.

Let τ be the set of rectilinear Steiner minimal trees for a terminal set T. Among the elements of τ, let τ_1 be the trees which maximize the sum of the degrees of the terminals. We characterize trees in τ_1 with the *leftness property*. A tree is said to have the leftness property if it is impossible to apply any of the following transformations: (1) a W-slide (West-slide), (2) a NW- or SW-flip, or (3) a N- or S-slide followed by transformation (2). Trees in τ_1 with the leftness property are called *optimal Steiner trees.* A *Steiner component* is a subtree of a tree in τ for which the sum of the terminal degrees equals the number of terminals in that subtree. An *optimal Steiner component* is a Steiner component of a tree in τ_1.

3. Review of previous theoretical results

We present in this section a review of theoretical results about RSMT. Those results will be used in the next section to demonstrate new properties of Hwang's *optimal Steiner tree* [7].

Theorem 3.0 [6]
The number of Steiner points of a RSMT is smaller or equal to n–2 where n is the number of terminal.

Theorem 3.1 [6]
There always exist a RSMT for some set of terminals T having the following property: every Steiner points are located at the intersection of a vertical and an horizontal lines passing through two terminals. The grid formed from all such vertical and horizontal lines is named *Hanan's grid.*

Theorem 3.2 [6]
Let have a Steiner point of degree 3 in a RSMT connecting points p_0, p_1 and p_2. Then no terminal lies in the smallest enclosing rectangle of p_0, p_1 and p_2.

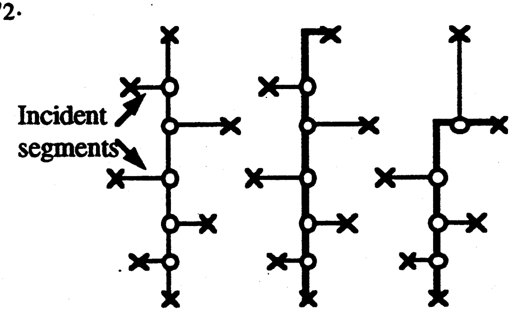


Figure 1

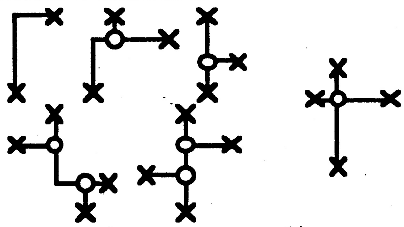
Theorem 3.4 [7,10]
"Let n>4. An optimal Steiner component for n terminals either consists of a single complete line with n–2 alternating incident segments, a complete corner with n–2 alternating segments incident to a single leg, or a complete corner with n–3 alternating segments incident to one leg and a single segment incident to the other leg (see fig. 1 where the thick lines are legs). For the corners, the segment closest to the corner node must point away from the opposite leg."
An optimal Steiner component for n terminals (n>4) has always one of the forms depicted in fig. 1 (crosses and circles are respectively terminals and Steiner points). For those cases, every Steiner point has a degree of 3.

Corollary 3.5 [10]
"Optimal Steiner trees consist of optimal Steiner components joined at terminals of degree 2, 3 or 4."

Theorem 3.6 [7]
Any RSMT may be reduced to an optimal Steiner tree.

4. Characterization of optimal Steiner Trees

We make in this section a characterization of optimal Steiner trees (OST). We have extended theorem 3.4 for the case when $n \leq 4$ terminals.



a) b)

Figure 2

Corollary 4.0
Theorem 3.4 is applicable for any n when only Steiner points of degree 3 are present.

Proof
We present in fig. 2a an exhaustive list of every different component topology having Steiner point of degree 3 for $n \leq 4$. The corollary is always observed. □

Without proof, we present the following lemma:

Lemma 4.1
The only component's topology having a Steiner point of degree 4 is presented in fig. 2b. In that case, two terminals have the same x-coordinate and two others have the same y-coordinate. Furthermore, the intersection of the segments between those two pairs of terminals is not empty.

Corollary 4.2
Any Steiner point of an OST is connected both horizontally and vertically by one line to at least one terminal.

Proof
By examining exhaustively any OSC topology (fig. 1 and 2) we observe that this corollary is true for any of them. Since OST are composed of OSCs (by definition), the corollary is observed by any OST. □

We define the *body* of an optimal Steiner component as its complete line composed of the largest number of Steiner points. When an OSC have two complete lines composed of the same number of Steiner point (0 or 1), then the vertical one is selected arbitrarily as the body (fig. 2a). If an OSC is composed of a corner, than the line adjacent to this corner which is not a body is called an *arm*. An OSC that does not have a corner, does not have an arm. From theorem 3.4, we know that the body and the arm of an OSC are each terminated by at least one terminal.

4.1 Characterization of the form of an optimal Steiner component

Theorem 4.3

Any OSC have one of the 4 following orientations (a body is represented by a thick line):

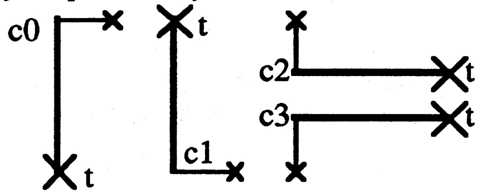
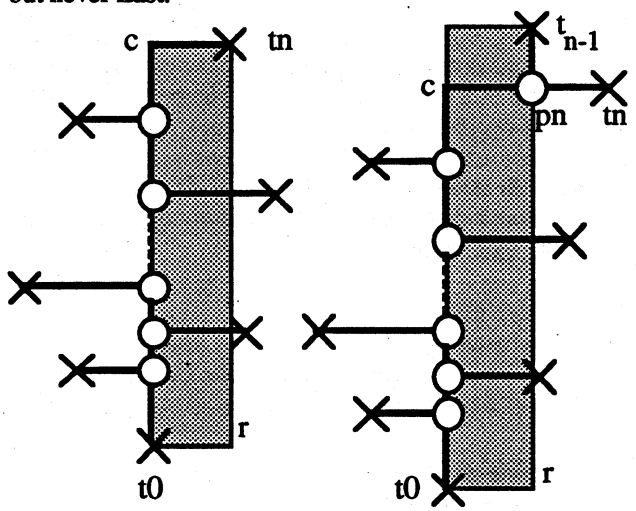


Figure 3

Partial proof

Any other orientation does not satisfy the leftness property. □

From theorem 4.3, a corner of an OSC can only have direction SE (corners c_0 or c_3) or NE (corners c_1 or c_2) but never SW nor NW. A body terminated by a terminal t may only have a direction relative to t (fig. 3): North (corner c_0), South (corner c_1) or West (corner c_2 or c_3) but never East.



a) Figure 4 b)

Theorem 4.4

A Steiner component with corner c having incident body and arm respectively terminated by terminals t_0 and t_n, can be an OSC if at least one of the following two cases is true:

a) the smallest enclosing rectangle r of terminals t_0 and t_n does not contain any terminal (fig. 4a);

b) the arm is composed of one Steiner point p_n which is connected by two segments to terminals t_{n-1} and t_n, and the smallest enclosing rectangle r of terminals t_0 and t_{n-1} does not contain any terminal (fig. 4b);

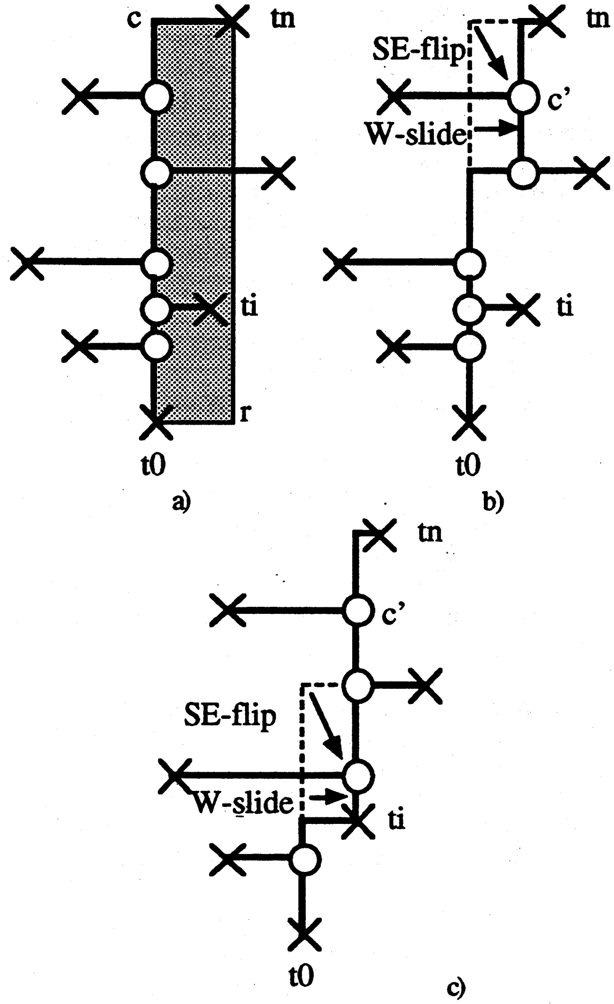


Figure 5

Proof

Consider case a). Suppose to the contrary that there exist at least one terminal in r and select the terminal t_i in r which is closest to the body. It is then possible to apply corner flips and segment slides to produce a new Steiner tree having the same length.

More precisely, let consider without lost of generality the configuration depicted in fig. 5a. Lets c' be a point having the same x-coordinate has t_i and the same y-coordinate as the Steiner point connected by a segment to c. We may apply a SE–flip to c to create a new corner c' and following this, we may apply a W-slide (fig. 5b). This double operation is reapply from top to bottom until t_i is reached. This is possible since the body has alternating incident segments. The newly found tree is depicted in fig. 5c) and it has the same length as the initial component by definitions of slide and flip operations.

We now have two cases: the degree t_i may be increased by 1 or be the same.

– If it is the same, it means that the initial component is member of an OST having a segment going North from t_i (not shown in the fig. 5a). If this is the case, we have now an OST of smaller length since we collapse two segments one over the other. Thus the initial component cannot be an OSC since it is not a member of an optimal Steiner tree.

– If the degree of t_i is increased, than by integrating the two newly found components to the original tree, we have a new tree having greater sum of the degrees of the terminals than the initial component. This is in contradiction with the definition of optimal Steiner tree (trees in τ_1 maximize the sum of the degrees of the terminals) and thus the initial component may not be an OSC.

The contradiction is completed, thus no terminal may lie in rectangle r for case a).

A similar proof may be done for case b). □

Lemma 4.5

Consider an OSC having two Steiner points p_i and p_j connected by a segment s. Those points (p_i, p_j) are respectively connected to terminals (t_i, tj) by segments perpendicular to s. Then the rectangle r having corners t_i and tj does not contain any terminal (fig. 6).

Proof

From theorem 3.2 implies that the smallest enclosing rectangle of two points connected to the same Steiner point must be empty. Thus the rectangles having corners (t_j, p_i) and (p_j, t_i) are empty. Since segment s does not contain any terminal in its relative interior, we are certain that no terminal lies in rectangle r. □

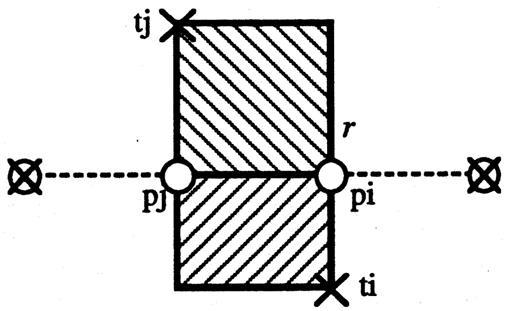


Figure 6

Let have a set of terminals t_0, t_1, ...,t_m and a set of non-terminal points $p_1, p_2,..., p_m$. Consider the following configuration:

1- terminal t_i ($1 \le i \le m$) has the same x-(y-)coordinate as point p_i;

2- all p_i are located *consecutively* on Hanan's grid on the same horizontal (vertical) line as t_0, and on the same side relative to t_0;

3- No terminal lies on the same horizontal (vertical) line as t_0.

4- every t_i ($1 \le i \le m$) is on the same side (left, right, top or bottom) relative to the p's line;

5- no terminal collinear to p_i,t_i lies on the other side of the p's line.

Such configuration with point (t_0, t_1,..., t_m, p_1,..., p_m.) is called a *cluster*. An example of a cluster having its t_i ($1 \le i \le m$) over an horizontal line passing through t_0 is depicted in fig. 7.

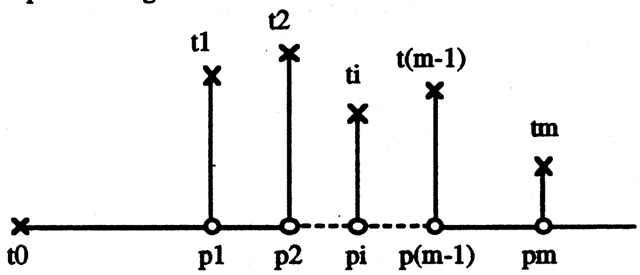


Figure 7

Corollary 4.6

At most one of the non-terminal p_i of a cluster (t_0, t_1,..., t_m, p_1,..., p_m.) may be a Steiner point of an OSC.

Proof

Firstly, lets note that no more than one non-terminal point among $p_1, p_2,..., p_m$ may be Steiner point of a body of an OSC. If this was not the case, we would have two connected Steiner points of a body with non-alternating segments which is not possible for an OSC.

We will now prove that if more than one of the non-terminal point of a cluster is a Steiner point, then they must be in the body of the same Steiner component and thus cannot compose an OSC. For the sake of clarity, we will consider without lost of generality the cluster depicted in fig. 7. From corollary 4.2 we know that if a non-terminal point is a Steiner point than it must be connected by a line to t_0. This must be the case since t_0 is the only terminal on its horizontal line (from cluster's definition). If two non-terminal points are Steiner points then they are both connected by a line to t_0. Since they are on the same side relative to the t_0, they must be member of the same body. □

Corollary 4.7

In a cluster (t_0, t_1,..., t_m, p_1,..., p_m.) only the p_i with terminal t_i closest to p's line may be a Steiner point of a body or an arm terminated by t_0.

Proof

We first consider the case of a body. From corollary 4.6 we know that in a cluster at most one non-terminal point may be selected as Steiner point of the same body. Let suppose to the contrary that a body have a Steiner point p_i having terminal t_i which is not the closest to p's line. Then we have two cases:

a) there is at least a terminal t_j closer to p's line in the direction of terminal t_0;

b) there is at least a terminal t_j closer to p's line in the direction opposite to terminal t_0.

For both of those cases we always have a violation of lemma 4.5:

In case a), point p_i must be connected in the direction of t_0 to a Steiner point p_k or t_0 itself (from theorem 3.4). For both of those cases the smallest enclosing rectangle of t_i and t_0, or t_i and pk does contain terminal t_j.

In case b), point p_i must be connected in the direction opposite to t_0 to a Steiner point p_k on p's line or a terminal t_k that is on the other side of p's line (from theorem 3.4). In both of those cases the smallest enclosing rectangle of t_i and tk or t_i and p_k does contain the terminal t_j.

The proof is similar for the case of an arm. □

4.2 Number of Steiner points and Optimal Steiner components

Let have an OST with T as its set of terminals, S and Q respectively as its set of Steiner points of degree 3 and 4, and finally C as its set of OSC.

Theorem 4.8

$$|C| + |S| + 2*|Q| = |T| - 1 \qquad \text{(eq. 1)}$$

Proof

The proof is done by induction. Firstly, note that equation 1 holds for any OSC (fig. 1 and 2). Secondly, let A_1 and A_2 be two subtrees of an OST having one common terminal. Define T_i, S_i, Q_i and C_i respectively as the set of terminals, Steiner points of degree 3 and 4, and OSC of A_i. We know from the induction step that

$$|C_1|+|S_1| + 2*|Q_1| = |T_1|-1$$
$$|C_2|+|S_2| + 2*|Q_2| = |T_2|-1$$

and by adding both equations

$$|C_1| + |S_1| + |C_2| + |S_2| + 2*(|Q_1|+|Q_2|) = |T_1| + |T_2| - 2 \qquad \text{(eq. 2)}$$

Since A_1 and A_2 are distinct except for one common terminal, we have

$$|T_1 \cup T_2| +1 = |T_1| + |T_2| \text{ and}$$
$$|C_1 \cup C_2| + |S_1 \cup S_2| + 2*|Q_1 \cup Q_2| = |C_1|+|C_2| +|S_1|+ |S_2| + 2*(|Q_1|+|Q_2|).$$

Consequently, by substituting from equation 2,

$$|C_1 \cup C_2| + |S_1 \cup S_2| + 2*|Q_1 \cup Q_2| = |T_1| + |T_2| - 2 = |T_1 \cup T_2| - 1$$

Thus, composed tree $(A_1 + A_2)$ observe equation 1.

□

The importance of the previous result is that it enables to determine the exact number of Steiner points in a OST depending on its number of OSC. Previous result from [6] (theorem 3.0) only give an upper bound on the number of Steiner points in a Steiner tree. In our case, an exact procedure can take into account the number of OSC to reduce accordingly the number of Steiner points in the searched domain. It is interesting to note that the maximal number of Steiner points, that is $|T| - 2$ (theorem 3.0) is attainable only and only if one component is present.

5. Enumerating Potential Steiner Points

It is possible to enumerate all potential Steiner points of an OST by the following procedure:

1- following theorem 4.3 and 4.4, determine every potential corner of an OST;

2- for each of those corners, compute a set of potential Steiner points following corollary 4.6 and 4.7.

We present in fig. 8 the results of the application of four procedures to reduce the number of potential Steiner points of a specific set of terminals (crosses and dots represent respectively terminals and potential Steiner points). We have in fig. 8a) the rectilinear convex hull [12] (47 points); in fig. 8b) Provan [9] result using Steiner hull (41 points); in fig. 8c) our new results (14 points); and in fig. 8d) the application of our result and Provan algorithm [9] (13 points). This example shows an interesting characteristic of our procedure that permits to use geometrical characterization to furthermore reduce the size of the potential Steiner point set. It would be easy to prove that the potential Steiner points found by our procedure are always member of the rectilinear convex hull of the terminal point set.

A careful analysis of the special case of fig. 7d permits to determine that no component may have more than 2 Steiner points. Furthermore, since at most one component may have more than 1 Steiner point in an OST (with further analysis), we are able to find from theorem 4.8 a bound of 5 (instead of 8 from theorem 3.0) on the number of Steiner points.

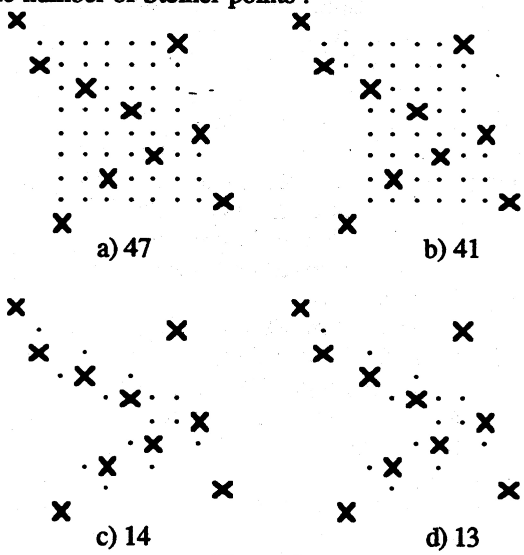


Figure 8

6. Conclusion

We have presented in this paper a topological characterization of Hwang's optimal Steiner trees. This characterization permits to determine configuration yielding optimal Steiner components, and this only by considering the topology of terminals' position. We presented an equation defining exactly the number of Steiner points of an OST following its number of terminals and optimal Steiner components. Finally, we presented a simple procedure to compute the set of points that can be Steiner points of an OST for some set of terminals. One interesting property of this procedure is that it only relies on topological relation of the set of terminals.

The characterization of optimal Steiner trees should permits the design of more efficient algorithms for obtaining exact or approximate solutions to this problem. Also, it can be used to find interesting special cases of the rectilinear Steiner minimal tree problem.

Acknowledgement

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