Balanced Cuts of a Set of Hyperrectangles*

(Extended Abstract)

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Abstract

We are given a set of n d-dimensional (possibly intersecting) isothetic hyperrectangles. The topic of this paper is the separation of these rectangles by means of a cutting isothetic hyperplane. Thereby we assume that a rectangle which is intersected by the cutting plane is cut into two non-overlapping hyperrectangles. We investigate the behavior of several kinds of balancing functions, as well as their linear combination and present optimal algorithms for computing the corresponding balanced cuts in $O(d\,n)$ time and space.

1 Introduction

Given a set of n d-dimensional (possibly intersecting) isothetic hyperrectangles, we consider the problem of separating these rectangles by means of a cutting isothetic hyperplane. If the cutting plane crosses a rectangle, this is cut into two non-overlapping hyperrectangles. We present optimal algorithms for computing several kinds of "balanced cuts" in $O(d\,n)$ time and space. Thereby, the balance function can be defined in different ways, leading also to different optimal partitions. We consider two important balancing strategies and their linear combination. First, the balance of a partition is defined to be the maximum number of hyperrectangles on either side of the cutting hyperplane after the split. We call the cut that minimizes this number the so-called *best balanced cut*. In opposite to that, the second cutting objective is to minimize the *size* of partitions, i.e. the total number of rectangles in the two subspaces. However, this second balancing function is only useful when used in conjunction with some criterion that guarantees the balance of the partition. Finally, we investigate *optimal* partitions with respect to balancing functions which can be described as a linear combination of the two balancing functions above. Our results include a generalization of recent results by [1] and [3].

A solution to this cutting problems has a number of applications, e.g., in the context of binary space partitions, partition trees, computer graphics and solid modeling, VLSI design, computer cartography and GIS, as well as geometric divide-and-conquer algorithms in many variations (see, e.g., [2, 3, 4, 5, 6, 7, 8, 9]).

In [1], it was shown that there always exists a cutting hyperplane separating the set S of n d-dimensional non-overlapping isothetic hyperrectangles into two sets each containing at most $\lceil \frac{2d-1}{2d} n \rceil$ hyperrectangles. Later, this result was extended to the more general case of

*The authors acknowledge the support of the Swiss National Science Foundation SNF within the ESPRIT project AMUSING.

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overlapping rectangles by [3]. It was shown that there always exists a cutting plane which creates two halfspaces with at most $\alpha = \lceil \frac{2d-1}{2d}(n-k) \rceil + k$ rectangles on each side, under the assumption of *k-overlapping*, i.e. at most $(k+1)$ of the rectangles in S are allowed to intersect at a common point. Here, a problem is that the upper bound approaches n in the limit as d increases, thus becoming an overestimated upper bound on the balance in general (not in the worst case).

Using a different measure, namely the so-called *profile* k^* of the scene which is defined to be the maximum number of rectangles (minimized over all dimensions) that can be intersected by a isothetic hyperplane, we obtain tighter bounds for the separation function above. Using this new measure, we can prove the existence of a cutting plane which creates two halfspaces with at most $\lfloor \frac{n+k^*}{2} \rfloor$ rectangles on each side.

The paper is organized as follows. Section 2 introduces the necessary definitions. Afterwards, in Section 3, we investigate best balanced cuts with the new profile factor thus improving the results in [3]. Finally, Section 4 includes the second balancing function by considering linear combinations of two balancing functions that maximize the balance of a partition and minimize the number of intersected rectangles. Due to space restrictions, some proofs are omitted.

2 Preliminaries

At the beginning, we are given a set $S := \{R_1, \ldots, R_n\}$ of n (possibly overlapping) isothetic hyperrectangles[1]

$$R_i := [a_{1i}, b_{1i}] \times \ldots \times [a_{di}, b_{di}]$$

in d-dimensional real space $\mathbb{R}^d$. Any isothetic hyperplane C (which can be regarded as some degenerate isothetic rectangle) separates $\mathbb{R}^d$ into two closed half-spaces. Any rectangle R whose interior is intersected by C is split into two non-overlapping parts corresponding to the respective sides of C. More precisely, for any $c \in]a_j, b_j[$, the hyperplane $x_j = c$ splits the rectangle $R = [a_1, b_1] \times \ldots \times [a_d, b_d]$ into two parts: $R_{low} := [a_1, b_1] \times \ldots \times [a_j, c] \times \ldots \times [a_d, b_d]$ and $R_{high} := [a_1, b_1] \times \ldots \times [c, b_j] \times \ldots \times [a_d, b_d]$. So, any isothetic hyperplane C induces a partition of S in the sense that some rectangles lie entirely on some side of the hyperplane while others are intersected. Let $C^<$, $C^>$ denote the sets of rectangles lying entirely in one of the two halfspaces generated by the cut C, respectively, and let $C^=$ denote the set of rectangles intersected by C. Now, it is desirable to obtain a cutting plane such that

(i) the number of rectangles after the split is as small as possible, or equivalently, the number of intersected rectangles is minimized and

(ii) the difference of the numbers of rectangles on both sides of the cutting plane is minimized.

Formally, we define the *minimum intersection cut* as a cut minimizing the following sum:

$$\Sigma_C = |C^<| + |C^>| + 2 * |C^=|$$

where $|C|$ denotes the cardinality of the set C. Note that, since for an arbitrary cut C we have $|C^<| + |C^>| + |C^=| = n$, minimizing Σ_C is equivalent to minimizing the number of intersected

[1] For the sake of simplicity, we often refer to hyperrectangles (hyper-planes) simply as rectangles (planes respectively), implicitly assuming that we always talk about scenes in d-dimensional real space, except when it is explicitly stated otherwise.

rectangles $|C^=|$. However, this minimum intersection cut will also lead to degenerate cuts since $\Sigma_C = 0$ outside the scene. Thus this balance function is only useful when applied in conjunction with some other criterion that guarantees the balance of the partition.

In opposite to that, a *best balanced cut* is defined to be a cut which minimizes the difference between $C^<$ and $C^>$, i.e.

$$\Delta_C = \| \, |C^<| - |C^>| \, \|$$

Thereby, $\|x\|$ denotes the absolute value of x. In general, these two functions together cannot always be minimized simultaneously. So, we propose a compromise, that is to look for an *optimal cut* C which minimizes the weighted sum $\gamma\Sigma_C + \delta\Delta_C$ for real positive parameters γ and δ (see Section 4).

Let C_j be an arbitrary isothetic plane normal to the j-th coordinate direction. We denote by k_j^* the maximum number of rectangles intersected by any such cut C_j, i.e. $k_j^* = \max_{C_j} |C_j^=|$. With that, the *profile* k^* denotes the minimum over all k_j^*, i.e. $k^* = \min\{k_1^*, \dots k_d^*\}$. Let $\{x_1, \dots, x_m\}$ the ordered sequence of the elements of the multiset $\{a_{j1}, \dots, a_{jn}, b_{j1}, \dots, b_{jn}\}$ defined by the projections of the rectangles onto the j^{th} coordinate direction.[2] We abbreviate this set by $\{x_i\}_1^m$. For a cut C at coordinate x, let $left(x) = |C^<|$ and $right(x) = |C^>|$. Notice that unlike [1, 3] we do not assume the rectangles to be in general position.

3 The Best Balanced Cut

The first lemma of this section gives a general classification of the best balanced cut for non-general position scenes.

Lemma 1 *Given a set S of n possibly overlapping segments on a real line, let m denote the maximum number of coinciding end points. Then, we can always find an interval $(\bar{x}_1, \bar{x}_2)$ such that any cut C at $x \in (\bar{x}_1, \bar{x}_2)$ satisfies $\Delta_C \leq m/2$.*

Notice that in the case of disjoint endpoints, we obtain $m = 1$ and, with that, the existence of a sub-interval where $left(x) - right(x) \leq 1/2$ or equivalently $left(x) - right(x) = 0$. Our first theorem now provides an upper bound on the balance of the best balanced cut.

Theorem 1 *Given a set of n possibly overlapping isothetic rectangles in d-dimensional real space, there always exists a cutting plane which creates two halfspaces with at most $\alpha^* = \lfloor \frac{n+k^*}{2} \rfloor$ rectangles on each side.*

The following lemma relates the k-overlapping factor of a rectangular scene with the definition of the "profile" k^* of the scene.

Lemma 2 *Given a set of n k-overlapping rectangles in d-dimensional real space, then the following inequality holds:*

$$k \leq k^* \leq \frac{(d-1)n + k}{d}$$

[2] In the following, we consider all coordinate directions separately.

Using this, we prove that in general (not in the worst-case) the upper bound α^* is tighter than the upper bound α, i.e. we have

$$\alpha^* \leq \alpha$$

Indeed, it is easy to produce examples showing that in many cases α^* can be considerably smaller than α. Thus the profile of the scene entails a better bound on the balance that can be obtained by the best balanced cut.

The proof of Theorem 1 gives rise to an $O(d\,n)$ algorithm to compute the best balanced cut C or, more precisely, an interval which contains all best balanced cuts. (Note that there may be infinitely many cuts which satisfy the balance condition). To do this, we have to find a sub-interval I where Δ_I is minimal. Under the general position assumption, we obtain $\Delta_I = 0$ in which case a simple $O(d\,n)$ algorithm for the best balanced cut has been found (cf. [3]). Notice that this algorithm can be easily extended to non-general position scenes by computing the median of the multiset of projected endpoints.

Theorem 2 *Given a set of n possibly overlapping isothetic rectangles in d-dimensional real space, a best balanced cut can be computed in optimal $O(d\,n)$ time and space.*

A more careful investigation of the problem illustrates that the best balanced cut may intersect a large number of rectangles thus creating an large size of the partition which is not desirable in practice. To overcome this disadvantage, we discuss in the following section another balance function which takes into account the balance and the size of the partition.

4 The Optimal Cut

In this section, we consider the question of satisfying the balance- and the minimum intersection condition, as defined in Section 2, at the same time. In general, a solution that is optimal in both respects does not exist. In other words, one can not always obtain an optimal cut C_O such that for all other cuts C, we have $\Delta_{C_O} \leq \Delta_C$ and $\Sigma_{C_O} \leq \Sigma_C$. Nevertheless, we can make a compromise between these two conflicting goals by considering the weighted sum of both balancing functions. Formally, we define the *optimal cut* as the one minimizing the following weighted sum

$$\Sigma^*_C = \gamma \Sigma_C + \delta \Delta_C$$

Thereby, γ and δ denote real positive parameters, with $\gamma + \delta = 1$ without loss of generality.

Let us first investigate the case of $0 \leq \gamma \leq 1/2$, i.e. $\delta \geq \gamma$. In other words, we favor the balance condition over the minimum intersection condition. For that, we look at the function Σ^*_C in more detail.

$$\Sigma^*_C \;=\; \gamma\,[|C^<| + |C^>| + 2|C^=|] \;+\; \delta\,\big||C^<| - |C^>|\big| \qquad (1)$$

Now, we distinguish two cases:

Case 1 : $|C^<| \geq |C^>|$

Substituting $\delta = 1 - \gamma$ in equation (1) and using the fact that $|C^<| + |C^>| + |C^=| = n$, we obtain:

$$\Sigma_C^* = 2\gamma n + (1 - 2\gamma)|C^<| - |C^>|$$

As $2\gamma n$ is constant and $(1 - 2\gamma)|C^<| - |C^>|$ is monotonously increasing for increasing x-values of the cut C, with $0 \leq \gamma \leq 1/2$, Σ_C^* is minimized if C is taken at the left end of the domain $|C^<| \geq |C^>|$. More precisely, it may happen that the condition $|C^<| \geq |C^>|$ creates an open interval. In that case, we take a cut which lies arbitrarily close to the left end of the interval.

Case 2 : $|C^<| \leq |C^>|$

Analogously, equation (1) can be reduced to

$$\Sigma_C^* = 2\gamma n + (1 - 2\gamma)|C^>| - |C^<|$$

Again $2\gamma n$ is constant and $(1-2\gamma)|C^>|-|C^<|$ is monotonously decreasing for increasing x-values of the cut C, with $0 \leq \gamma \leq 1/2$. Thus, Σ_C^* is minimized if C is taken at the right end of the domain $|C^<| \leq |C^>|$.

In both cases, we see that the minimization of Σ_C^* is equivalent to the computation of the interval that minimizes $\||C^<| - |C^>|\|$. After that, one of the "endpoints" of this interval minimizes the function Σ_C^*.

Let us now discuss what happens in the case of $1/2 \leq \gamma < 1$, i.e. $\delta \leq \gamma$, where the minimum intersection condition dominates the balance condition.[3] In this case, we obtain non-monotonous functions Σ_C^*. In order to compute a minimum of the function Σ_C^*, we first sort the coordinates of the endpoints in ascending order (for each dimension separately) and compute the entire function Σ_C^*. This can be done using $O(d\,n \log n)$ time and $O(d\,n)$ space. Up to now, it is an open problem if this running time can be improved.

Theorem 3 *Given a set of n possibly overlapping isothetic rectangles in d-dimensional real space, an optimal cut with respect to the weighted sum $\Sigma_C^* = \gamma\Sigma_C + \delta\Delta_C$, for $\gamma \leq \delta$, can be found in $O(d\,n)$ time and space. In the case of $\gamma > \delta$, the function Σ_C^* can be minimized in $O(d\,n \log n)$ time and $O(d\,n)$ space.*

5 Conclusion and Generalizations

Our approach can be generalized to more general classes of objects, e.g. convex objects. In this case, our algorithm and all investigations of the best achievable balance apply to the *bounding boxes* of the objects, as well. All we have to demand to preserve our runtime and space bounds is that the boundaries of the bounding boxes of the objects can be computed in $O(d)$ time, each. In contrast, in our general case, no corresponding results are known if we drop the restriction that the cutting hyperplanes be isothetic.

[3]Notice that in the case of $\gamma = 1$ and $\delta = 0$ an "optimal solution" consists of a cut outside the scene which can be found in $O(n)$ time.

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