The Minimum Cooperative Guards Problem on k-Spiral Polygons
Extended Abstract

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ABSTRACT We propose a new variation of the art gallery problem. We first define the guards visibility graph in which all vertices represent guards and there is an edge between two guards if and only if these two guards can see each other. Our minimum cooperative guards problem requires that the resulting guards visibility graph of the solution must be connected and of course the number of guards is minimized. We show that this problem is NP-hard for general polygons. We also propose two linear algorithms for solving this problem on k-spiral polygons, for $k = 1$ and 2.

1. Introduction

The ***art gallery problem*** is to find the placement of a minimum number of guards in an art gallery such that every point in the gallery can be visible from at least one guard. The art gallery is represented by a polygon and the guards are stationary points in the polygon. It is known that the art gallery problem is NP-hard [LL 86]. Many variations of the art gallery problem consider mobile guards, such as mobile guards that patrol along an edge, diagonal, or arbitrary line segment of the given polygon [O'R 87], the ***watchman route problem*** [CN 88, CN 91], the ***m-watchmen route problem*** [NW 90, MW 91, CNN 91], the ***two-guard walkability problem*** [IK 91, T 93], and the ***k-guard walkability problem*** [T 93]. The variations of the art gallery problem and its results can be found in [O'R 87, Sh 92].

In this paper we propose a new variation of the art gallery problem. In contrast to the allowance of a guard to patrol along a route, we introduce the relation of "cooperation" between stationary guards in the art gallery problem. The motivation of defining this new problem is as follows: It is rather dangerous for a guard to be stationed inside, say an art gallery, if he is not watched by some of his companions. We define a ***guards visibility graph*** $GVG(A, P)$ on a set A of guards in a polygon P as follows: the vertex set is A and there is an edge between two guards if and only if they are visible to each other in P. In addition to finding a set A of minimum number of guards that can see the given polygon P, we require that $GVG(A, P)$ is connected. We call the guards in our problem the ***cooperative guards*** and call this problem the ***minimum cooperative guards problem***, abbreviated as the ***MCG problem***.

The MCG problem is proved in outline to be NP-hard for simple polygons in the next section. We present algorithms for solving the problem on a restricted class of polygons, called k-spiral polygons, for $k = 1$ and 2. In Section 3, we present a linear time algorithm for 1-spiral polygons. Moreover, we also solve the ***constrained*** MCG problem for 1-spiral polygons in Section 4. The constrained version of the problem is the same as the original one except that a specified point must be included in the solution. For 2-spiral polygons, we partition it into three subpolygons in which two of them are 1-spiral polygons. By "matching" the guards in these two 1-spiral subpolygons such that they can be visible to each other and then solving the constrained MCG problems on each of them, we obtain a linear algorithm to solve the MCG problem on 2-spiral polygons and present it in Section 5. Finally, we give concluding remarks in Section 6. Limited by space, we omit all proofs. Interested readers may consult [LHL 93].

2. The NP-Hardness of Minimum Cooperative Guards Problem

The NP-hardness of the MCG problem immediately follows the proof in [LL 86] establishing the NP-hardness of art gallery problem. In [LL 86], an instance of the Boolean three satisfiability problem (3SAT) is reduced to the minimum vertex guard problem for simple polygons. There is a set A of minimum number of guards stationed in the transformed simple polygon P if and only if the instance of 3SAT is satisfiable. It can be seen that $GVG(A, P)$ is connected due to the transformation itself.

3. The MCG Problem for 1-Spiral Polygons

A ***1-spiral polygon*** P is a simple polygon whose boundary can be partitioned into a reflex chain RC and a convex chain CC [NW 90]. Traversing the boundary of P counterclockwise, the starting (ending) vertex of RC is called v_s (v_e). Starting from v_s (v_e), we draw a line along the first (last) edge of RC, until it hits the boundary of P at l_1 (r_1). This line segment $v_s l_1$ ($v_e r_1$) and the first (last) part of CC starting from v_s (v_e) form a region, called the ***starting (ending) region.*** Fig. 3-1 shows an example. Note that there must be a guard stationed in both starting and ending regions.

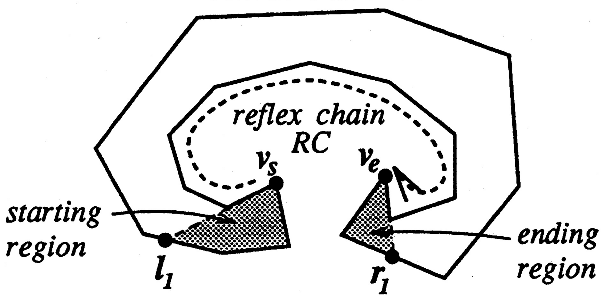


Fig. 3-1 A 1-spiral polygon.

Let us define some new terms. Let a be a point in a polygon P. Given a reflex chain RC, we can draw two tangents with respect to RC from a. If the exterior of RC lies entirely on the right-hand (left-hand) side of a tangent drawing from a, we call it the ***left (right) tangent*** of a with respective to RC. Draw the left (right) tangent of a with respect to RC until it hits the boundary of P at b. We call ab a ***left (right) supporting line segment*** with respect to a and call b the ***ending point.***

We present the greedy algorithm, called $MCG1_e$, as follows. The algorithm starts by stationing a guard at l_1. Then we find a point l_2 on the convex chain such that $l_1 l_2$ is a left supporting line segment with respect to l_1. If l_2 is in the ending region, then we are done. Otherwise, we repeat the process until the ending point of our newly created left supporting line segment is in the ending region. See Fig. 3-2.

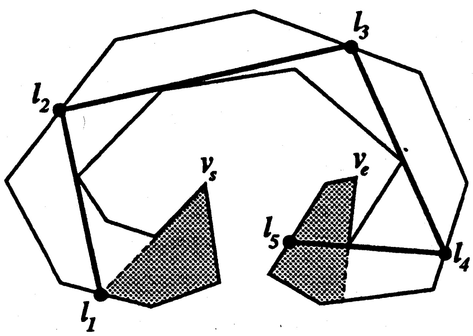


Fig. 3-2 A set of minimum cooperative guards $\{l_1, l_2, l_3, l_4, l_5\}$ in a 1-spiral polygon.

Theorem 3.1 Algorithm $MCG1_e(P)$ is optimal for the MCG problem on a 1-spiral polygon P.

Corollary 3.1 Algorithm $MCG1_e$ runs in linear time.

Before going further, for solving the MCG problem for 1-spiral polygons, note that by the symmetry of the starting and the ending region, we can start from r_1 and successively find the right supporting line segments until the ending point of the last one is in the starting region. We call this counterpart algorithm Algorithm $MCG1_s(P)$. The subscripts e and s distinguish these two algorithms.

4. The Constrained MCG Problem for 1-Spiral Polygons

The ***constrained MCG problem*** is the same as the MCG problem except that a specified point must be included in the solution. We first slightly modify Algorithm $MCG1_e(P)$ ($MCG1_s(P)$) to obtain Procedure $CG1_e(P, a)$ ($CG1_s(P, a)$), where a is a point in P. Procedure $CG1_e(P, a)$ ($CG1_s(P, a)$) is similar to Algorithm $MCG1_e(P)$ ($MCG1_s(P)$) except that the starting point l_1 (r_1) is replaced by a. Procedure $CG1_e(P, a)$ ($CG1_s(P, a)$) outputs a set of points $\{a=a_1, a_2, ..., a_h\}$ and returns the value h such that $a_i a_{i+1}$ forms a left (right) supporting line segment, for $1 \le i \le h-1$, and a_h is in the ending (starting) region.

Next, we define two specific regions as follows. Before doing so, we have to define some notations which will be used throughout the rest of this paper. For a 1-spiral polygon P, k denotes the minimum number of cooperative guards in P, and $\{l_1, l_2, ..., l_k\}$ and $\{r_1, r_2, ..., r_k\}$ are two sets of points resulting from Algorithms $MCG1_e(P)$ and $MCG1_s(P)$, respectively. A subchain of the boundary of P from point p to point q in counterclockwise order is denoted as $C[p, q]$. The subpolygon visible from a point p in P is denoted as $VP(p)$. Let l_0 be v_s and r_0 be v_e. We define L_i to be the region bounded by $C[l_i, l_{i-1}]$ and $l_{i-1}l_i$, but exclusive of l_{i-1}, for $1 \le i \le k$. We define R_i similarly to be the region bounded by $C[r_{i-1}, r_i]$ and $r_i r_{i-1}$ but exclusive of r_{i-1}, for $1 \le i \le k$.

Now we are ready to present the algorithm, called ***CMCG1(P, a)***, where a is in a 1-spiral polygon P. In $CMCG1(P, a)$, we first execute Algorithms $MCG1_e(P)$ and $MCG1_s(P)$ to obtain two sets of points $\{l_1, l_2, ..., l_k\}$ and $\{r_1, r_2, ..., r_k\}$, respectively. Next, we test whether a is in $L_i \cap R_{k-i+1}$, for some i, or not. If a is in $L_i \cap R_{k-i+1}$ for some i, we perform Procedures $CG1_e(P, a)$ and $CG1_s(P, a)$ and report the combined results; otherwise, we simply report $\{a, l_1, l_2, ..., l_k\}$. Fig. 4-1 shows two examples.

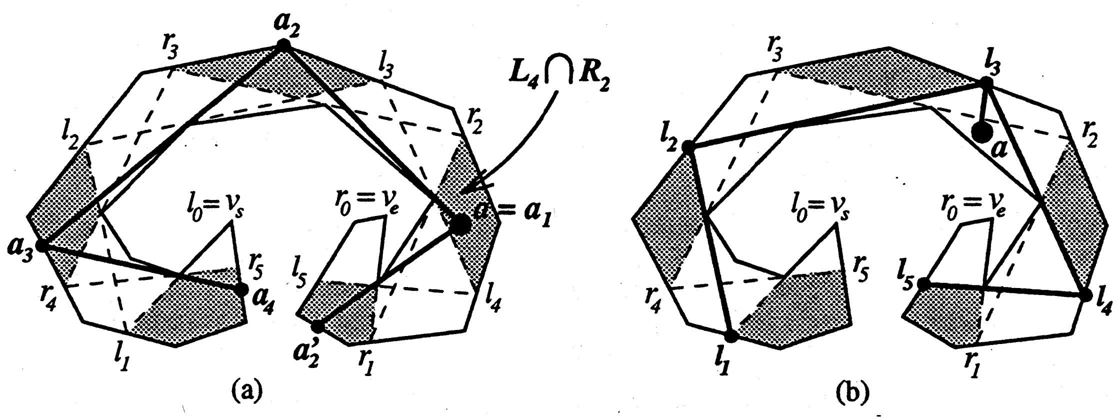


Fig. 4-1 Examples for illustrating Algorithm $CMCG1(P, a)$.

Lemma 4.1 (1) $CG1_s(P, a) = h$ if point a is in L_h, where $1 \le h \le k$.
(2) $CG1_e(P, a) = h$ if point a is in R_h, where $1 \le h \le k$.

Theorem 4.1 For a 1-spiral polygon P, if a is in $L_i \cap R_{k-i+1}$ of P for some i, Algorithm $CMCG1(P, a)$ reports k cooperative guards; otherwise, Algorithm $CMCG1(P, a)$ reports $k+1$ cooperative guards.

Theorem 4.2 Algorithm $CMCG1(P, a)$ is optimal for the constrained MCG problem on a 1-spiral polygons P with a specified point a in P.

Corollary 4.1 Algorithm $CMCG1(P, a)$ runs in linear time.

5. The MCG Problem for 2-Spiral Polygons

A ***2-spiral polygon*** P is a simple polygon whose boundary can be partitioned into two reflex chains and two convex chains. Traversing the boundary of P counterclockwise, the reflex chains and the convex chains are encountered alternatively. We label these four chains as RC_1, CC_1, RC_2, and CC_2.

5.1 An Algorithm *MCG2* to Solve the MCG Problem for 2-Spiral Polygons

The Algorithm $MCG2$ consists of the following three steps:

Step 1: Divide a 2-spiral polygon *P* into three subpolygons: *U*, *D*, and *M*.

The algorithm is based upon the observation that we can divide a 2-spiral polygon into three specific subpolygons. Consider the 2-spiral polygon in Fig. 5-1(a). For RC_1 and RC_2 of the 2-spiral polygon, we can draw two outer common tangents $u_1'u_2'$ and $d_1'd_2'$. As can be seen in Fig. 5-1(a), the 2-spiral polygon is partitioned into three subpolygons by two line segment u_1u_2 and d_1d_2 induced from these two outer common tangents. In general, two of them are 1-spiral polygons, denoted as U and D, and the other one is denoted as M. There are degenerated cases. For example, the subpolygon D may degenerate into a convex polygon, a line segment, and a point, as shown in Figs. 5-1(b), (c), and (d), respectively. The discussion of the degenerated cases will be omitted in this extended abstract. In the following, we assume that the 2-spiral polygon consists of two 1-spiral subpolygons U and D.

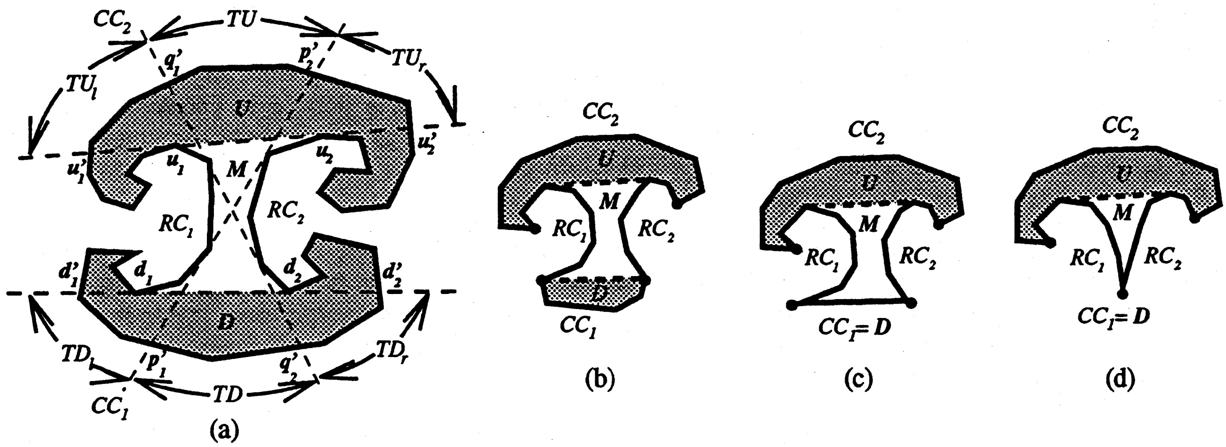


Fig. 5-1 Three subpolygons U, D, and M in 2-spiral polygons.

Step 2: Find TU and TD in U and D such that we can place two guards in them respectively and these two guards are visible to each other.

We identify the parts of CC_1 and CC_2 in which a guard stationed per each of it are possibly visible to each other by finding two inner common tangents $p_1'p_2'$ and $q_1'q_2'$. Consider Fig. 5-1(a) again. We denote $C[p_2', q_1']$ as TU and $C[p_1', q_2']$ as TD. In CC_2, $C(q_1', u_1')$ is denoted as TU_l and $C(u_2', p_2')$ is denoted as TU_r. In CC_1, we denote $C(d_1', p_1')$ as TD_l and $C(q_2', d_2')$ as TD_r. Note that we use brackets (parentheses) to emphasize the inclusive (exclusive) of the end points of the chain.

Since we shall station the guards on CC_1 and CC_2, let us define two specific subchains of the convex chain in a 1-spiral polygon: one is $C[l_i, r_{k-i+1}]$, $1 \le i \le k$, and the other is $C(r_{k-i}, l_i)$, $0 \le i \le k$. We shall use bold line segments to denote $C[l_i, r_{k-i+1}]$ and call it a ***bold subchain***, and dashed line segments to denote $C(r_{k-i}, l_i)$ and call it a ***dashed subchain***.

Step 3: "Match" a guard in U with a guard in D and find the final solution.

There are two distinct cases:

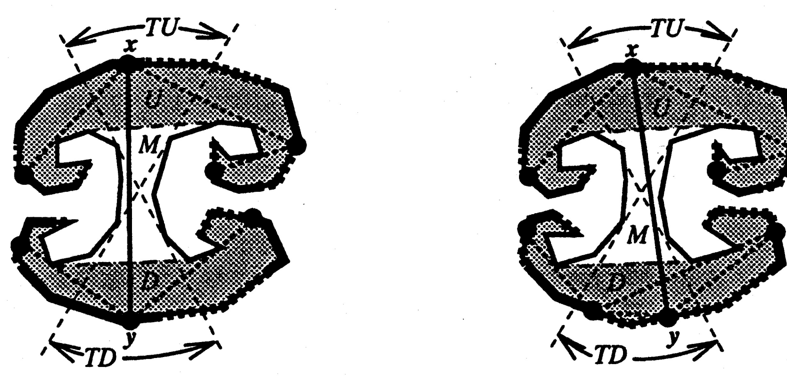
Case 1: There are bold subchains in TU or TD, or both. Again, there are two subcases:

Case 1.1: There are an x in a bold subchain of TU and a y in a bold subchain of TD such that x and y are visible to each other, as shown in Fig. 5-2(a).

In this case, we perform Algorithms *CMCG1*(U, x) and *CMCG1*(D, y).

Case 1.2: For any pair of bold subchains BS_u and BS_d of TU and TD, respectively, there is no x in BS_u and no y in BS_d such that x and y are visible to each other, as shown in Fig. 5-2(b).

In this case, assume that there is a bold subchain BS_u in TU. We pick any x in BS_u and then find a corresponding y in a dashed subchain of TD such that x and y are visible to each other. Then, we perform Algorithms *CMCG1*(U, x) and *CMCG1*(D, y).



(a) Case 1.1 (b) Case 1.2

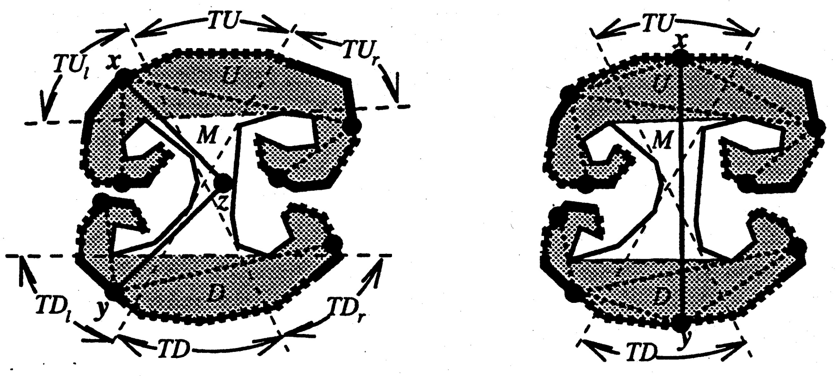
Fig. 5-2 Illustrations of Case 1 in Step 3.

Case 2: There are bold subchains in neither TU nor TD. Again, there are two subcases:
Case 2.1: There is a z in M such that z is visible to an x in a bold subchain of TU_l (TU_r) and a y in a bold subchain of TD_l (TD_r), as shown in Fig. 5-3(a).

In this case, we perform Algorithms *CMCG1*(U, x) and *CMCG1*(D, y).

Case 2.2: For any pair of bold subchains BS_u and BS_d of TU_l (TU_r) and TD_l (TD_r), respectively, there is no z in M such that z is visible to an x in BS_u and a y in BS_d, as shown in Fig. 5-3(b).

In this case, we pick any x in TU and then find a corresponding y in TD such that x and y are visible to each other. Then, we perform Algorithms *CMCG1*(U, x) and *CMCG1*(D, y).



(a) Case 2.1 (b) Case 2.2

Fig. 5-3 Illustrations of Case 2 in Step 3.

Let $g(P)$ denote the number of minimum cooperative guards of the MCG problem for a 1-spiral polygon P. Following Theorem 4.1 directly, the following two properties are satisfied:
(1) Algorithm *CMCG1*(P, a) reports a set of $g(P)$ cooperative guards if a is in a bold subchain.
(2) Algorithm *CMCG1*(P, a) reports a set of $g(P)+1$ cooperative guards if a is in a dashed subchain.

The results of Algorithm *MCG2* is summarized in the following table.

Cases	Case 1.1	Case 1.2	Case 2.1	Case 2.2
number of guards	$g(U)+g(D)$	$g(U)+g(D)+1$	$g(U)+g(D)+1$	$g(U)+g(D)+2$

5.2. The Correctness and Analysis of Algorithm *MCG2*

Let QU be the region bounded by $C[u_2', u_1']$ and $u_1'u_2'$, but exclusive of u_1' or u_2'. Let QD be the region bounded by $C[d_1', d_2']$ and $d_2'd_1'$, but exclusive of d_1' or d_2'. In QU and QD, let RU and RD be two regions bounded between two inner common tangents, respectively. See Fig. 5-4.

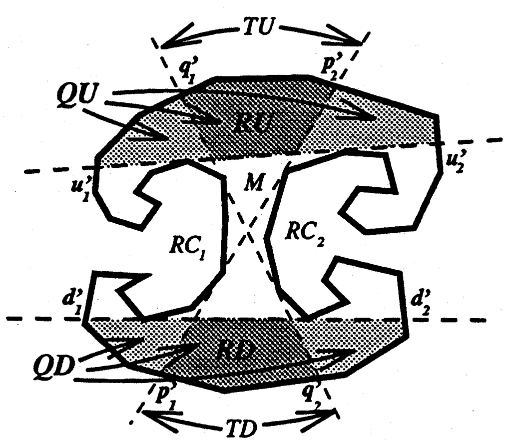


Fig. 5-4 The regions QU, RU, QD, and RD.

Let A be any set of (not necessarily minimum) N cooperative guards which can see the entire P.

Lemma 5.1 There is at least one guard of A stationed in each of QU and QD.

Lemma 5.2 $N \geq g(U) + g(D)$.

Sketch of the proof: Let a be outside of U. Then, $VP(a) \cap U \subseteq VP(x) \cap U$ where x is in QU. This means that a can be removed without altering the visibility of guards in U. Thus, stationing a guard outside of U can not have the effect of reducing the number of minimum cooperative guards needed for U. The number of cooperative guards in U (D) is at least $g(U)$ ($g(D)$). Q. E. D.

Lemma 5.3 Let a be in QU and b be in QD. If a is not in RU or b is not in RD, then a and b can not be visible to each other.

Theorem 5.1 Algorithm *MCG2* is optimal for the MCG problem on 2-spiral polygons.
Sketch of the proof: Let H be a set of cooperative guards resulting from Algorithm *MCG2*. The connectivity of $GVG(H, P)$ is due to the algorithm itself. It can be proved that the visibility of H can cover P. We discuss the minimality of H as follows. According to Lemma 5.1, let two guards a and b be in QU and QD, respectively. We discuss various cases which depending upon whether a is in RU or not and whether b is in RD or not. In addition to $g(U)+g(D)$ cooperative guards, according to Lemma 5.3, we determine one or two more guards are needed in each case in Step 3. Q. E. D.

Lemma 5.4 There are at most two bold subchains in TU (TD). If there is no bold subchain in TU (TD), there is exactly one bold subchain in TU_l (TD_l), or TU_r (TD_r), or both.

The analysis of Algorithm *MCG2* is as follows. In step 1, finding outer common tangents in a 2-spiral polygon P can be solved in linear time [LHL 93]. In step 2, finding inner common tangents in M of P takes $O(\log n)$ time [GHS 91]. In Step 3, since there are constant number of bold subchains which must be tested in each case according to Lemma 5.4, and each test can be done in linear time, Step 3 is linear.

Corollary 5.1 Algorithm *MCG2* runs in linear time.

6. Concluding Remarks

We have proposed a new variation of the art gallery problem, called the minimum cooperative guards problem. The problem is proved to be NP-hard for simple polygons. We presented two linear time algorithms for solving the minimum cooperative guards problem on 1-spiral and 2-spiral polygons, respectively. We have also solved the constrained minimum cooperative guards problem on 1-spiral polygons in linear time.

For this new problem, much remains to be studied. For example, is there a polynomial algorithm for 3-spiral polygons, or other k-spiral polygon, where $k > 3$? Is there a polynomial algorithm for other constrained class of polygons?

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