Delaunay Triangulations and Computational Fluid Dynamics Meshes

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Abstract

In aerospace computational fluid dynamics (CFD) calculations, the Delaunay triangulation of suitable quadrilateral meshes can lead to unsuitable triangulated meshes. In this paper, we present case studies which illustrate the limitations of using structured grid generation methods which produce points in a curvilinear coordinate system for subsequent triangulations for CFD applications. We discuss conditions under which meshes of quadrilateral elements may not produce a Delaunay triangulation suitable for CFD calculations, particularly with regard to high aspect ratio, skewed quadrilateral elements.

1 Introduction

In computational fluid dynamics (CFD) applications, the problem domain must be discretized into meshes (or grids) over which the governing equations of fluid dynamics are solved. The two major classes of grids for aerospace CFD applications are structured grids and unstructured grids. Structured grids are curvilinear grids designed so that the neighbors to any element are implicitly known. These grids have been studied for quite some time and techniques for their construction are well understood [2]. Unstructured grids are composed of elements in which neighbors must be explicitly listed. The component elements are usually, but not necessarily, triangular. These grids are currently not as widely used in aerospace applications, and have been the object of recent interest [3, 4, 5]. Several properties of Delaunay triangulation [6, 7, 8, 9] make it attractive to use in unstructured grid generation. However, a major drawback to this method is the need for a separate method of point generation; a straightforward approach to this drawback is the use of a structured grid generator to create the necessary points [10, 11, 12, 13].

Certain features of structured grids are useful to maintain in unstructured grids. Most commonly, structured grids are body-fitted curvilinear meshes where contours follow the object boundaries. Structured grids may also contain very high aspect ratio elements. This allows properties of the problems being solved to be exploited; in general, very high solution gradients can exist perpendicular to surfaces, and very small solution gradients tangent to surfaces. Unstructured meshes should exhibit the same "structure" which orients the cells along a feature of interest. When a structured grid contains high aspect ratio grid cells, the Delaunay triangulation of the grid points may not maintain the original grid lines of the structured grid (Figures 1 and 2). Orientation of the cells in a particular direction is lost when this occurs.

This study investigates conditions under which Delaunay triangulations of points from structured grids will maintain the original grid structure. The effects of skew and aspect ratio on the Delaunay triangulation are studied.

2 Preliminaries

First consider a general case of points distributed arbitrarily along straight lines a distance h apart, with the points no more than s apart, and $s > h$. What relationship must exist between s and h so that the lines are guaranteed to be in the Delaunay triangulation of the points.

Let $P_1 = (-s/2, 0)$ and $P_2 = (s/2, 0)$ be two points on the line $y = 0$ (the results can be generalized to any location, but these points were chosen for simplicity of analysis). One scenario could have points on successive contours shifted by $s/2$, e.g., $R_1 = (0, h)$

and $R_2 = (0, -h)$. For the edge P_1P_2 to be included in the triangulation of the points, the circumcircle for P_1, P_2, R_1 must not contain R_2, and likewise, the circumcircle for P_1, P_2, R_2 must not contain R_1. The circumcenter of P_1, P_2, R_1 is $(0, h/2 - s^2/8h)$. The circumcenter of P_1, P_2, R_2 is $(0, -h/2 + s^2/8h)$. To guarantee that edge P_1P_2 is in the triangulation, then

$$\begin{aligned} -h/2 + s^2/8h &< h/2 - s^2/8h \\ s^2/h^2 &< 4 \\ s/h &< 2 \end{aligned}$$

assuming distances positive.

Without knowing anything else about the distribution of points along the contour lines, we must conclude that the aspect ratio of the triangles can be no greater than 2 to guarantee that all original edges are in the triangulation. The introduction of structure into the points locations allows for larger aspect ratios under certain conditions. The remainder of this section defines some of the concepts to be used, and following sections explore this idea in further detail.

Definition 1 *A* structured grid *is an undirected graph G=(V,E) with vertex set V and edge set E such that*

$$\begin{aligned} V &= \{v_{i,j} \mid \forall v_{i,j} \exists (x_{i,j}, y_{i,j}) \in \Re^2, \\ &\qquad 0 \le i \le m, 0 \le j \le n\} \\ E &= \{(u, w) \mid \forall v_{i,j} \in V, \\ &\qquad i \ne m \Rightarrow (v_{i,j}, v_{i+1,j}) \in E \\ &\qquad j \ne n \Rightarrow (v_{i,j}, v_{i,j+1}) \in E\} \end{aligned}$$

Definition 2 *A graph $G = (V, E)$ is called* Delaunay-embeddable *if G is a subgraph of the Delaunay triangulation of V, $D(V)$.*

Definition 3 *A* rectangular structured grid *is a structured grid $G = (V, E)$ where the location of each $v_{i,j}$ is given by:*

$$\begin{aligned} x_{i,j} &= i \cdot s \\ y_{i,j} &= j \cdot h \end{aligned}$$

where $s, h \in \Re$, $0 \le i \le m$, and $0 \le j \le n$.

Points in a rectangular structured grid have a constant spacing in x and a (possibly different) constant spacing in y.

Definition 4 *The* aspect ratio *of a structured grid element is the ratio s/h, where s is the length of a long side and h is the separation between the sides of length s.*

Definition 5 *A* skewed structured grid *with aspect ratio s/h is a structured grid where the location of each $v_{i,j}$ is given by:*

$$\begin{aligned} x_{i,j} &= i \cdot s + j \cdot dx \\ y_{i,j} &= j \cdot h \end{aligned}$$

where $dx \le s/2$, $0 \le i \le m$, and $0 \le j \le n$.

Points in a skewed structured grid still exhibit constant spacing in x and y, however, at each level, a constant shift in x from the previous level results in a grid of parallelograms. The amount of skew in a skewed structured grid is θ, the angle formed by a perpendicular to one of the quadrilateral sides at a corner (Figure 3).

Definition 6 *The* Delaunay angle cut-off *is the angle beyond which a skewed structured grid is no longer Delaunay-embeddable.*

Definition 7 *A* simple stretched structured grid *is a structured grid where the location of each $v_{i,j}$ is given by:*

$$\begin{aligned} x_{i,j} &= i \cdot s + \sum_{k=0}^{j} dx(1+\epsilon)^k \\ y_{i,j} &= \begin{cases} 0, & j = 0 \\ \sum_{k=0}^{j-1} h(1+\epsilon)^k, & j = 1, 2, \ldots, n \end{cases} \end{aligned}$$

where $dx \le s/2$, $0 < \epsilon < s$, $0 \le i \le m$, and $0 \le j \le n$.

The simple stretched structured grid is one which exhibits a constant multiplicative growth in spacing between levels. This will be referred to as a stretched structured grid for the remainder of the paper.

The following facts are to be noted: The center of any circle which passes through $P_1 = (-s/2, 0)$ and $P_2 = (s/2, 0)$ will lie on the line $x = 0$ (in general, the center will lie on the line which is the perpendicular bisector to line setment P_1P_2). Delaunay triangulations have the following properties: For any convex quadrilateral in the triangulation, the diagonal is selected such that the minimum angle is maximized; the diagonal selected is not necessarily the shortest diagonal. For the degenerate case of four co-circular points, one of two edges may be selected; in such cases, the edge which is a member of the edge set E is chosen.

3 Skewness and Aspect Ratio

Given a set of quadrilaterals, ideally the principle direction of the skewed triangles should follow the original boundaries of the quadrilaterals. However, if there is skew (i.e., the quadrilaterals are parallelograms rather than rectangles), then it is possible for the Delaunay triangulation to break the quadrilateral boundaries. This section describes the conditions under which this happens.

The case of a skewed structured grid with $s > h$ is studied (Figure 4). The goal is to produce a Delaunay triangulation of the points in the vertex set V such that each triangle lies between the lines $y = jh$ and $y = (j+1)h$. In other words, we will determine the restrictions on the grid such that the skewed structured grid is Delaunay-embeddable.

First let us consider any parallelogram with aspect ratio s/h, with $s > h$. The short diagonal creates two angles, α and β, with β opposite the side of length s. As the parallelogram is skewed by θ, an amount δ is added to two opposing corners and subtracted from the remaining two corners, causing a shift of dx to the upper two corner points (Figure 5).

Lemma 1 *As θ increases, α and β both increase.*

Proof: Consider points P_3 and P_4 at the upper corners of a rectangle. Shift these points a distance dx relative to P_1 and P_2. This is the same as adding δ to angle $P_1P_3P_4$ and angle $P_1P_2P_4$. Originally, the diagonal from P_3 to P_2 made angles α and β with the sides P_1P_2 and P_2P_4. Since we started with a rectangle, α and β are guaranteed to be less than 90 degrees. Since $s > h$, $|P_3P_2| > |P_1P_3|$. When the points P_3 and P_4 are shifted a distance dx to new points P_3' and P_4', two new angles α' and β' are achieved. Let θ_1=the angle $P_3P_2P_3'$ and θ_2=the angle $P_4P_2P_4'$. $\alpha' = \alpha + \theta_1$ and $\beta' = \beta - \theta_1 + \theta_2$. It is immediately apparent that $\alpha' > \alpha$. $\beta' > \beta$ because $|P_3P_2| > |P_1P_3|$ implies $\theta_2 > \theta_1$ while $dx \leq s/2$. □

Now consider adjacent quadrilaterals $P_1P_2P_3P_4$ and $P_3P_4P_5P_6$ within a skewed structured grid. A skewed structured grid is Delaunay-embeddable for the degenerate case $dx = 0$ since the corner points of any quadrilateral will be co-circular. For $P_1P_2P_3P_4$, either P_1P_4 or P_2P_3 could be chosen as diagonals. The following facts are to be noted:

- $dx = 0$ and $s > h \Rightarrow \beta < 90$ and $\angle P_1P_2P_3 > \angle P_3P_2P_5$.
- $\angle P_1P_2P_3 \equiv \angle P_4P_3P_2$
- As a quadrilateral $P_1P_2P_3P_4$ is skewed, edge P_2P_3 will be in the Delaunay triangulation of its points.
- $\angle P_1P_2P_3$ is the smallest angle in the Delaunay triangulation of $P_1P_2P_3P_4$.

Lemma 2 *If $\beta < 90$ degrees, then a skewed structured grid is Delaunay-embeddable.*

Proof: From Lemma 1, we know that β increases as dx increases. It follows that β approaches 90 degrees as dx increases. Since the lines are a constant distance h apart, when β=90 degrees, the diagonal line P_3P_2 becomes a perpendicular bisector for the line P_1P_5, and at this point P_3P_2 also bisects the angle formed by $P_1P_2P_5$.

The convex quadrilateral formed by points $P_2P_3P_5P_4$ (Figure 6) has as its diagonal either P_3P_4 or P_2P_5. When $dx = 0$, we know that P_3P_4 is selected as the diagonal, because $\angle P_4P_3P_2$ is the larger of the smallest angles in the possible triangulations of this quadrilateral. As a skew of θ is introduced to the grid, P_5 moves faster than P_3, so $\angle P_3P_2P_5$ grows faster than $\angle P_1P_2P_3$ (from lemma 1 we know that both will increase). When $\beta = 90$ degrees, the two angles are equal, and because $s > h$, $\angle P_4P_3P_2 < \angle P_5P_3P_4$. When β exceeds 90 degrees, then because $\angle P_3P_2P_5$ is increasing faster, it becomes the larger of the smallest angles in the possible Delaunay triangulations of $P_2P_3P_5P_4$, and P_2P_5 is then selected as the diagonal. □

When this occurs, the Delaunay triangulation no longer includes the original quadrilateral boundaries.

Lemma 3 *If $s/h \leq 2$, then a skewed structured grid is Delaunay-embeddable.*

Proof: At $dx = 0$, $\beta < 90$ degrees. β reaches a maximum at $dx = s/2$. When $s/h < 2$, $\beta < 90$ degrees at $dx = s/2$, and the grid remains Delaunay-embeddable. When $s/h = 2$, $\beta = 90$ degrees at $dx = s/2$, and the grid is Delaunay-embeddable. □

Theorem 1 *As aspect ratio increases, the Delaunay angle cut-off for a skewed structured grid decreases.*

Proof: From Lemma 2, we know that the Delaunay angle cut-off occurs when $\beta =$ 90 degrees. At this point, the Pythagorean theorem gives

$$dx = \frac{s - \sqrt{s^2 - 4h^2}}{2}$$

The Delaunay angle cut-off θ^* is defined by

$$\begin{aligned} tan\theta^* &= \frac{dx}{h} \\ &= \frac{s - \sqrt{s^2 - 4h^2}}{2h} \\ &= \frac{1}{2}\left(\frac{s}{h} - \sqrt{\frac{s^2}{h^2} - 4}\right) \end{aligned}$$

For $s/h \leq 2$, there is no Delaunay angle cut-off (lemma 3). As s/h increases, θ^* decreases (Figure 7). □

Therefore, whether or not a structured grid is Delaunay-embeddable is dependent on aspect ratio, and as aspect ratio increases, the "tolerance" for skew decreases. As s/h gets larger, the value of the Delaunay angle cut-off θ^*, where the Delaunay triangulation no longer includes the original quadrilateral boundaries, approaches 0.

For the case of a monotonic stretched structured grid, the skew angle derived above is a lower bound.

Theorem 2 *As aspect ratio increases, the Delaunay cut-off angle of a stretched structured grid decreases.*

Proof: We need only consider two adjacent quadrilaterals at a time from the stretched structured grid with aspect ratios s/h and $s/h(1+\epsilon)$. The skew angle θ can be derived from the circumcircles for the two interior triangles, and is defined by

$$\begin{aligned} tan\theta &= \frac{dx}{h} \\ &= \frac{s - \sqrt{s^2 - 4h^2 - 4\epsilon h^2 - \epsilon^2 h^2}}{(2+\epsilon)h} \\ &= \frac{1}{2+\epsilon}\left(\frac{s}{h} - \sqrt{\frac{s^2}{h^2} - 4 - \epsilon(4+\epsilon)}\right) \end{aligned}$$

Since successive levels have different (increasing) values of θ for ϵ positive, the grid will be Delaunay-embeddable when θ is based on the largest aspect ratio elements found in the grid. □

4 Conclusions

We have shown the limitations of Delaunay triangulations of points from structured grids for aerospace applications. For the general case of points distributed along fixed contours, we have shown a restriction on the aspect ratio for which Delaunay triangulations can be directly obtained. By imposing a structure on the point distribution, we have demonstrated the relationship between aspect ratio and quadrilateral element skew on the maintenance of contours from structured grids.

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Figure 1 – Portion of a CFD Grid

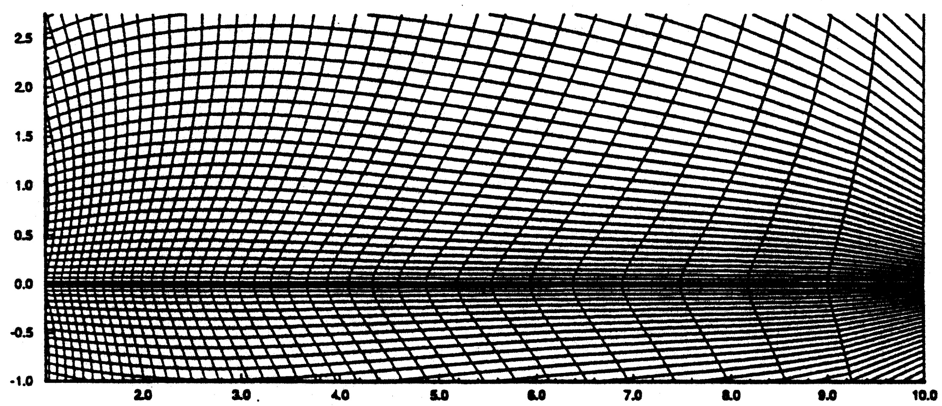
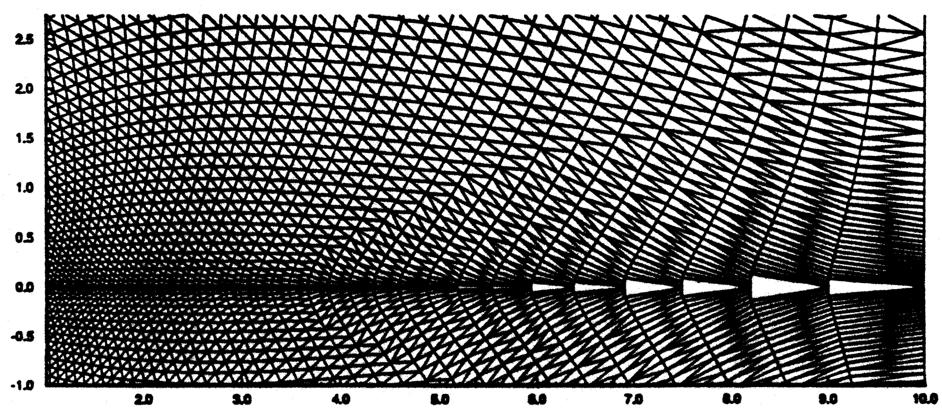


Figure 2 – Delaunay Triangulation of CFD Grid in Figure 1



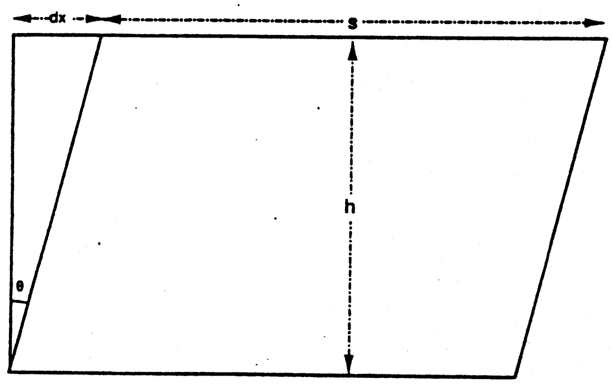


Figure 3 – A Skewed Structured Grid Element

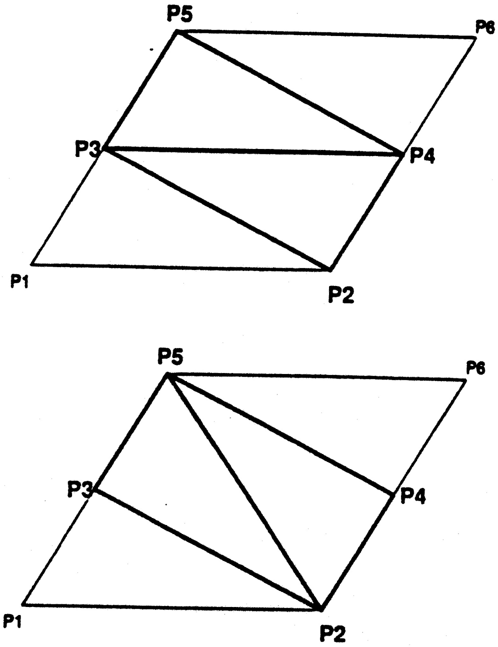


Figure 6 – β < 90 and β>90

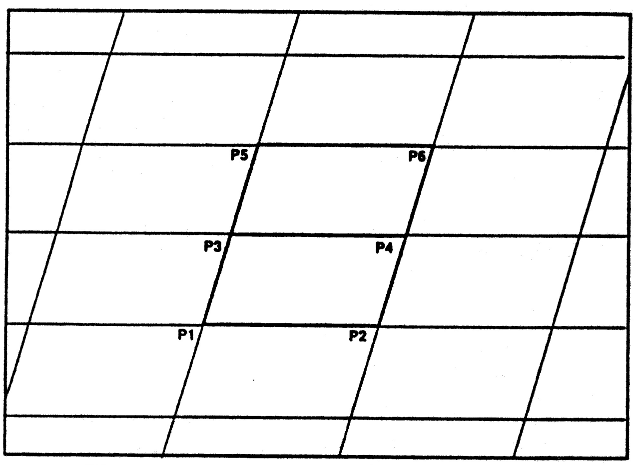


Figure 4 – A Skewed Structured Grid

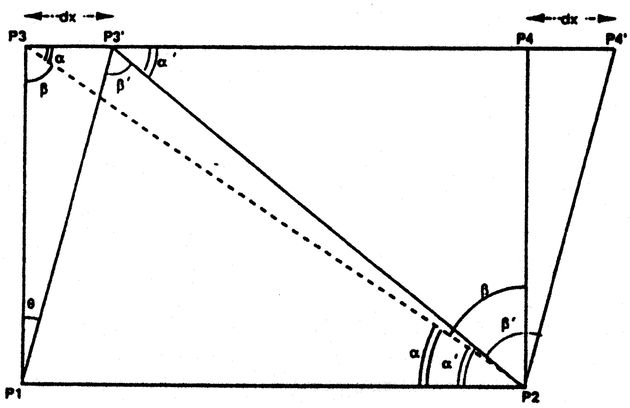


Figure 5 – Effects of Increasing Skew

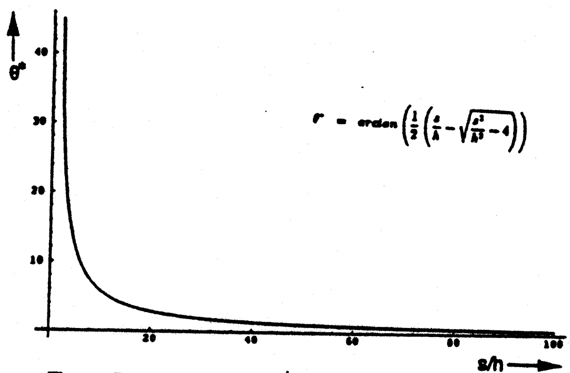


Figure 7 – Delaunay Angle Cut-Off vs. Aspect Ratio