# A Probably Fast, Provably Optimal Algorithm for Rectilinear Steiner Trees

**Extended Abstract**

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**Abstract**

We use the technique of divide-and-conquer to construct a minimal rectilinear Steiner tree on a set of sites in the plane. A well-known optimal algorithm for this problem by Dreyfus and Wagner [3] is used to solve the problem in the base case. The run time is probabilistic in nature: for all $\epsilon > 0$, there exists $b > 0$ such that Prob $(T(n) < 2^{b\sqrt{n}\log n}) > 1 - \epsilon$, for $n$ sites uniformly distributed on a rectangle. The key fact in the run-time argument is the existence of probable bounds on the number of edges of an optimal tree crossing our subdivision lines. We can test these bounds in low-degree polynomial time for any given set of sites.

## 1 Introduction

Steiner's Problem is this: given a set $V$ of points, called *sites*, in the Euclidean plane, find a set $S$ of points, called *Steiner points*, and a set $E$ of edges on $V \bigcup S$ such that $E$ forms a tree on $V \bigcup S$ and the sum of the lengths of the edges in $E$ is minimal. In this paper we focus on the rectilinear variant, where the distance between two points $(x_1, y_1)$ and $(x_2, y_2)$ is given by $d = |x_1 - x_2| + |y_1 - y_2|$. In the resulting *Rectilinear Steiner Problem*, edges are constrained to consist of vertical and horizontal segments only, and we again seek a tree of minimal length. We call such trees *rectilinear Steiner minimal trees*, or RSMTs for short. This problem is NP-complete [4]. Most research has thus focused on heuristic algorithms, sometimes with performance guarantees (e.g. [1]).

Two important theoretical results form the basis for our algorithm. Hanan [5] proves that the grid graph formed by drawing horizontal and vertical line segments through the sites always contains a minimal rectilinear Steiner tree. By restricting consideration to this grid graph, we can greatly reduce the search space for the problem. The second theoretical result is due to Hwang [6], who showed how every RSMT can be transformed into an equivalent tree in a canonical form. Such a canonical tree can be found in Hanan's grid graph. Moreover, any maximal line segment made up of edges in a canonical tree contains at least one site.

In this paper we use the technique of divide-and-conquer to construct an RSMT on a set of sites that is uniformly distributed in a rectangle. In Section 2 we describe our probably fast, provably optimal algorithm. This work is an extension of an earlier provably fast, probably optimal algorithm [8]. In this new algorithm the possible crossings of an optimal Steiner tree across the dividing line must be exhaustively searched. In Section 3 we are able to compute a probable bound on the size of the crossing set, which is used in restricting the search. In Section 4 the run time $T$ is shown to

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be probabilistic in nature: for all $\epsilon > 0$, there exists $b > 0$ such that

$$P(T < 2^{b\sqrt{n}\log n}) > 1 - \epsilon.$$

Informally, the run time is $2^{O(\sqrt{n}\log n)}$ with probability $1-\epsilon$, where the big-Oh constant is dependent upon $\epsilon$. The fastest previously-published optimal algorithm runs in time $O(n^2 3^n)$ [3]. Conclusions are given in Section 5.

## 2 An Optimal Divide-and-Conquer Algorithm

In this section we describe our basic divide-and-conquer algorithm. We assume that the set of sites is uniformly distributed in a rectangle $R$ with sides parallel to the coordinate axes. We will divide $R$ into two subrectangles $R_1$ and $R_2$ by a line segments $l$ perpendicular to $R$'s longest dimension, say $x$, through the site $p$ with median $s$-coordinate. This placement of $l$ ensures that the sites in $R_1$ and $R_2$ are uniformly distributed [2, page 18].

Our problem here is to generate a set of rectilinear Steiner trees on the sites containing at least one RSMT and to choose the shortest. Figure 1 shows a subdivided RSMT.

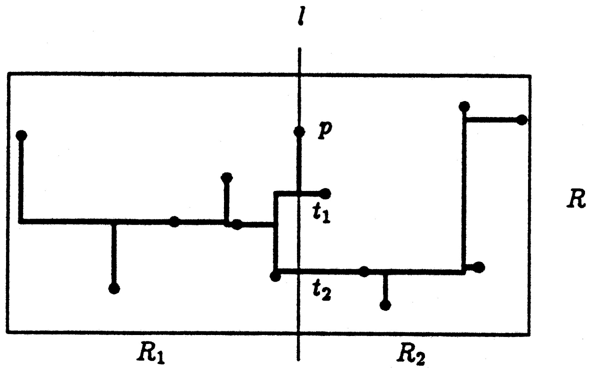


Figure 1: An optimal rectilinear Steiner tree subdivided.

The tree crosses $l$ in two points, $t_1$ and $t_2$, called *terminals*. When we remove the points along $l$, the Steiner tree falls apart into a forest, with one subtree in $R_1$ and two in $R_2$. The horizontal edges through $t_1$ and $t_2$ connect subtrees across $l$, while the edge $\overline{t_1 p}$ connects $p$ to the rest of the tree. In searching for solutions, therefore, we must consider all possible sets $C$ of terminals along $l$ and all possible sets $E$ of edges along $l$. We can restrict the locations of the terminals to points where $l$ crosses Hanan's grid graph. We must ensure that the two solutions will fit together to form a tree. For example, if a solution on $R_1$ connects $t_1$ and $t_2$ , then an optimal solution on $R_2$ will not connect $t_1$ and $t_2$ since it would form a cycle. Similarly, if $R_1$ does not connect $t_1$ and $t_2$, then $R_2$ must connect them. Thus, for each subproblem we provide information about connections external to the subproblem by representing external connections as an equivalence relation on the set of terminals.

In the more general subproblem that occurs after multiple subdivisions, we can have terminals around the entire boundary of $R$; we treat this set as vertices of a planar graph $G$ consisting of a rectilinear Steiner tree on a set of sites minus the interior of $R$. The components of $G$ determine an equivalence relation on $T$: two points in $T$ are equivalent if and only if they lie in the same component of $G$. Extending terminology from the graph theory literature, we call this an *outerplanar equivalence relation*. In Figure 2 we designate the equivalence relation on $T$ by labels on the terminals: two terminals with the same label belong to the same equivalence class. To solve this general subproblem we again divide $R$ in half with a line segment $l$ through midpoint $p$. We let $C$ be the set of all points on $l$ that might serve as terminals for the subproblems. $C$ includes the intersections of $l$ with all perpendicular bisectors from sites in $S$ to $l$. These are naturally a part of Hanan's grid and include all sites on $R \bigcap l$. Any other edge in a canonical solution to the problem on $R$ that passes through $l$ must pass through a site outside of $R$ [6]. Since sites outside of $R$ connect up to the solution inside $R$ through the terminals in $T$, the other points in $\mathcal{C}$ are the intersections of $l$ with all perpendicular bisectors from terminals in $T$. See Figure 2.

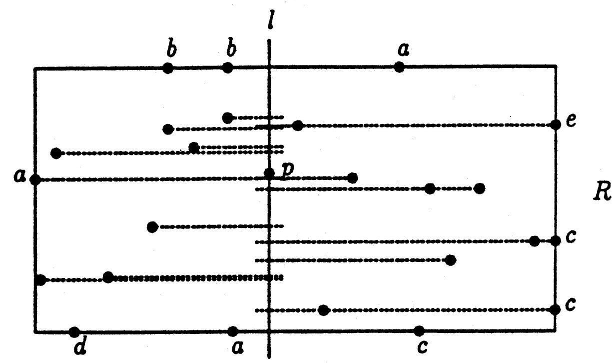


Figure 2: Subdividing the general subproblem.

What is the subproblem to be solved on $R_1$? The terminals for the subproblem consist of the set $T_1 \bigcup C$, where $T_1$ is the set of terminals in $T$ lying to the left of $l$ and $C$ is the subset of $\mathcal{C}$ currently under consideration. $C$ must always be chosen to contain $p$ and any other sites on $R \bigcap l$. We will show in Section 3 that $|C| \leq maxC(R)$, where $maxC(R)$ is a value that is easily computed from the coordinates of the sites in $R$. This restriction will yield a probabilistic improvement to the run time while maintaining the optimality of the result. The sites for the subproblem are those in the set $S_1$ of sites in $S$ lying to the left of $l$. An equivalence relation on $T_1 \bigcup C$ is partially determined by the equivalence relation on $T$ that is inherited by $T_1$ and by the set of edges $E$ on $C$, but this is not enough to ensure uniqueness. In fact, we must iterate over all outerplanar equivalence relations on $T_1 \bigcup C$ that are consistent with the equivalence relation on $T$ and the edge set $E$. For a given choice of $C$, $E$, and $eq_1$, the solution we seek for the $R_1$ subproblem is a minimal-length rectilinear Steiner tree $ST_1$ on $S_1 \bigcup T_1 \bigcup C$ assuming that terminals in the same equivalence class of $eq_1$ are connected by edges of length zero.

What is the subproblem to be solved on $R_2$? Let $S_2$ be the sites in $S$ to the right of $l$ and $T_2$ be the terminals in $T$ to the right of $l$. For a given choice of $C$, $E$, and $eq_1$, we have found a solution $ST_1$ to the left subproblem. This solution, together with the equivalence relation on $T$ uniquely determines an equivalence relation $eq_2$ on $T_2 \bigcup C$. Thus, we seek a minimal-length rectilinear Steiner tree $ST_2$ on $S_2 \bigcup T_2 \bigcup C$ assuming that terminals in the same equivalence class of $eq_2$ are connected by edges of length zero.

We now paste together the solutions $ST_1$ and $ST_2$ for the two subproblems to get a solution, $ST_1 \bigcup ST_2 \bigcup E$, to the subproblem for $R$. After iterating over all subsets $C$ of $\mathcal{C}$ with $|C| \leq maxC(R)$ and containing all sites on $R \bigcap l$, all edge sets $E$ on $C$, and all allowable equivalence relations $eq_1$ on $T_1 \bigcup C$, we choose the shortest solution for $R$.

The solution in the base case will be provided by Dreyfus and Wagner's $O(n^2 3^n)$ algorithm [3]. In this case the algorithm is run on Hanan's grid graph on $T \bigcup S$ with equivalent terminals connected by edges of length zero. The base case is reached when the number of sites in a subproblem drops below $2\sqrt{n}$.

# 3 Improving the Run Time of the Algorithm

In this section we describe Hwang's [6] canonical form for RSMTs, and we use it to derive a bound $maxC(R)$ on the number of edges in a canonical RSMT that cross a line that divides a rectangle

$R$ into two subrectangles. This bound depends on the number of sites in a strip to one side or the other of the line. Then we compute probabilities that $maxC(R)$ exceeds a given value, using the distribution of the number of sites within the strip.

For an RSMT $T$ on a set of sites $P$, a *Hwang component* of $T$ is a maximal connected subset $C \subseteq T$ such that there are no sites in the interior of $C$. We can obtain the Hwang components of $T$ as the closures of the connected components of the forest $T \setminus P$.

In an RSMT a *line* is a straight line segment contained within a single Hwang component. An *edge* is a portion of a line with no interior Steiner points that is terminated at each end at a site, a Steiner point, or a corner. An *equivalence operation* on a RSMT consists of either flipping a corner or sliding an edge that connects parallel lines in the RSMT. It may happen that a sequence of equivalence operations on a component $C$ results in splitting $C$ in two. If this is possible, we say that $C$ is *reducible*; otherwise $C$ is *irreducible*. The characterization of irreducible Hwang components is a central part of Hwang's work [6]. There it is shown that each such tree is equivalent to one in which all but possibly one of the Steiner points lie on a straight line that contains one of the sites. The one exceptional Steiner point joins to one end of the line of Steiner points through a corner. The edges attached to any line alternate sides. We may paraphrase Hwang's result as follows:

**Lemma 3.1** *Every rectilinear Steiner minimal tree is equivalent to a canonical tree, which can be decomposed into lines, each terminating at a site. Within a Hwang component, the edges attached to a line alternate sides.*

To bound the number of edges in a canonical RSMT that cross a vertical or horizontal line segment, we will need the following lemma.

**Lemma 3.2** *Let $T$ be a canonical RSMT on a set of sites. Let $U$ be a rectangle such that no edges of $T$ lie along the boundaries of $U$. We can associate edges $e$ entering the right side of $U$ with edge-disjoint paths $P(e)$ in $T$. A given path passes through the top, bottom, or left side of $U$, or it terminates at a site in $U$. At most one path passes through the top (bottom) of $U$. At most three paths terminate at the same site.*

**Proof Sketch:** The proof is constructive. We use the edge-alternation property of Hwang components to match edges to nearly-unique sites. The top and bottom limits are argued by contradiction of the optimality of $T$.

The following lemma can be used to bound the number of edges in a canonical RSMT that cross a vertical or horizontal line segment.

**Lemma 3.3** *Let $T$ be a canonical RSMT on a set of sites. Let $U$ be a rectangle with width $w$ and height $h$ such that no edges of $T$ lie along the boundaries of $U$. If $v$ is the number of sites in $U$ and $e$ is the number of edges of $T$ that pass through the right side of $U$, then $e \leq 3v + \frac{2h}{w} + 4$.*

**Proof Sketch:** We construct a graph $G$ connecting all sites and terminals of $T$. We bound $L(G)$, the length of $G$, with $L(T) \leq L(G) \leq L(T) + 2h + 4w + 3vw - ew$.

We are now ready to determine the bound $maxC(R)$ on the crossing edges for a rectangle $R$.

**Theorem 3.4** *Let $R$ be a rectangle containing $n'$ sites that arises as a subproblem in the computation of a RSMT on a set of sites $S$. Let $h$ be the height of $R$, and let $l$ be the length of $R$. Without loss of generality we may suppose that $l > h$. Let $p$ be the site in $R$ with median $x$-coordinate and let $L$ be the vertical line segment through $p$, terminated at the top and bottom boundary of $R$. Let $v$ be the number of sites to the left of $L$ within distance $h/\sqrt{n'}$. Let $T$ be a canonical RSMT on $S$. Then the number of edges of $T$ that cross $L$ from either side is no more than $maxC(R) = 6v + 4\sqrt{n'} + 11$.*

**Proof Sketch:** We construct two overlapping $h \times h/\sqrt{n'}$ rectangles to bound the number of edges entering the left (right) side of $L$.

We note that the values of $maxC(R)$ at the various levels of the algorithm do not depend on anything but the rectangles and the distribution of sites. Thus we can precompute all of the $maxC$s. Also, since the base case of the recursion occurs when $n' \leq 2\sqrt{n}$, there are at most $\sqrt{n}$ values of $maxC$ to be computed.

**Theorem 3.5** *Let $S$ be a set of $n$ sites uniformly distributed in a rectangle. Suppose the rectangle is partitioned into subrectangles for the divide-and-conquer algorithm. Then for any $\epsilon > 0$, there is a number $a(\epsilon)$ such that the probability is less than $\epsilon$ that the value of $maxC(R)$ exceeds $a(\epsilon)\sqrt{n'}$ for at least one subrectangle $R$ with $n'$ sites. Here, $a(\epsilon)$ does not depend on $n$.*

**Proof Sketch:** We bound the number of points in the $h \times h/\sqrt{n'}$ rectangles of Theorem 3.4.

# 4 Run-Time Analysis

We let $T(n)$ be the total run time required for *ProbTree* to find a minimal length Steiner tree on a set of sites $S$ with probability $1 - \epsilon$, where $n = |S|$. We will assume throughout the remainder of this section that the bound in Theorem 3.4 is satisfied. The proofs are omitted due to length constraints.

The following lemma gives a useful decomposition of $T(n)$. For this decomposition we define $T_M(n)$ to be the portion of the computation time $T(n)$ spent excluding the calls to Dreyfus and Wagner's algorithm, while $T_{DW}(n)$ defines the computation time spent in Dreyfus and Wagner's algorithm.

**Lemma 4.1** $T(n) = T_M(n) + T_{DW}(n)$ *and* $T_M(n) = o(T_{DW}(n))$.

**Corollary 4.2** $T(n) = O(T_{DW}(n))$.

Corollary 4.2 has reduced our task to that of counting the number of calls to the Dreyfus and Wagner algorithm. The first lemma gives a bound on the number of subsets $C$ that must be considered.

**Lemma 4.3** *Let $C$ be a set with $c > 1$ elements. The number of subsets of $C$ having $k$ or fewer elements is bounded above by $(c+1)^k/k!$.*

The next lemma gives a bound on the number of outerplanar equivalence relations that must be considered.

**Lemma 4.4** *Let $R$ be a rectangle and $T$ be a set of $t$ terminals on the boundary of $R$. Then the number of outerplanar equivalence relations $C_t$ on $T$ is bounded above by $4^t$.*

**Proof Idea:** We show that $C_t$ is the $t^{th}$ Catalan number as defined by Knuth in [7], which is bounded by $4^t$.

Now we let $N(n_i, t_i)$ be the number of calls made to the Dreyfus and Wagner algorithm by our algorithm in the process of solving a subproblem at level $i$ with at most $n_i$ sites and at most $t_i$ terminals. Because the stopping condition for the recursion is $n_d \leq 2\sqrt{n}$, then $N(n_i, t_i)$ depends implicitly on $n$. Moreover, $t_i$ can be bounded in terms of the $maxC$s of Theorem 3.4. The next lemma gives us a recurrence relation on $N(n_i, t_i)$.

**Lemma 4.5** *If $maxC(R) \leq a\sqrt{n_i}$ for all subrectangles $R$ encountered at level $i$ of the recursion, then*

$$N(n_i, t_i) \leq (n_i + t_i + 1)^{a\sqrt{n_i}} 4^{t_i + a\sqrt{n_i}} N(n_{i+1}, t_{i+1}).$$

Next we use the recurrence relation to get a bound on $N(n, 0)$.

**Lemma 4.6** *There exists $B > 0$ depending upon $a$ such that*

$$N(n, 0) < 2^{B\sqrt{n}\log n}.$$

Combining Theorem 3.5, Lemma 4.6 and Corollary 4.2, we now have the final result.

**Theorem 4.7** *For all $\epsilon > 0$ there exists $b > 0$ such that*

$$Prob(T(n) < 2^{b\sqrt{n}\log n}) > 1 - \epsilon.$$

## 5 Conclusions

We have described an optimal algorithm for rectilinear Steiner tree minimization that, with probability $1-\epsilon$ for $\epsilon > 0$, has a lower asymptotic run-time than any previously-published algorithm. Preliminary investigation shows, however, that our new algorithm is not yet ready for practical implementation. Its enumeration is inefficient, compared to the Dreyfus and Wagner approach, for all feasibly-small $n$. We are now trying to reduce the constant factor in the exponent of our algorithm's $2^{O(\sqrt{n}\log n)}$ run-time. For example, we believe that we might be able to make a factor-of-three improvement in our upper bound on the $maxC$s, the maximum number of new terminals in a subproblem. Since our run-time estimates are straightforward, we may be able to extend our probabilistic run-time guarantee to other input distributions.

We would like to emphasize that it is not difficult to compute the run-time of our algorithm, for a given set of sites, in advance of the computation. Such a run-time estimate could be done in low-order polynomial time, by counting the number of sites in the vicinity of the cuts in our recursive planar dissection. This indicates another possibility for constant-factor improvements in our algorithm. For a given set of sites, it may be better to subdivide at something other than the median coordinate. The tradeoff is between having an even split of the sites (i.e., a lower recursion depth), and having a small number of sites in the vicinity of a cut (i.e., a lower $maxC$).

In previous work, we explored a related divide-and-conquer algorithm for rectilinear Steiner tree minimization [8]. That algorithm was provably fast, but probably optimal if the sites were drawn from a uniform distribution. In essence, we asserted likely bounds on $maxC$. However, we subdivided *between* medial sites, rather than *through* the medial site. This may have some constant-factor advantages, but our proofs became very complicated due to the conditioning this scheme placed upon the site distribution within subdivisions.

We foresee some eventual practical applications for our work. In VLSI design, signals are typically distributed on wiring that takes the form of rectilinear trees. Minimum-length trees are low-capacitance trees, beneficial because they present low load to the signal driver. Steiner tree minimization algorithms are thus used in many systems for wire routing in VLSI. Also, the midline of the VLSI chip is sometimes a bottleneck for intrachip wiring. Our bounds on $maxC$, and our lemmas on the structure of canonical Steiner trees, may thus influence the design of future Steiner tree heuristics.

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