Improved algorithms for computing the shortest watchtower of polyhedral terrains

Binhai Zhu

School of Computer Science, McGill University, Montreal, Canada H3A 2A7

binhai@hesn.cs.mcgill.ca

Abstract

Given a polyhedral terrain S with n vertices, the shortest watchtower problem is defined to compute the shortest vertical line segment uv whose lower endpoint u lies on S and whose upper endpoint v can see the entire terrain. Sharir [Sha88] gave an $O(nlog^2n)$ time algorithm for solving the problem and posed the open problem of computing the shortest watchtower in $O(nlogn)$ time. In this paper, we show that by using Dobkin-Kirkpatrick's hierarchical representation of a convex polyhedron, the problem can be solved in $O(nlogn)$ time. This settles the above open problem posed by Sharir. For the generalized version of the problem, i.e., computing the shortest vertical distance between two non-convex polyhedral terrains, we give an $O(nlogn+k)$ time algorithm. Though k could be $O(n^2)$, in some cases it will bring the total time complexity down to $O(nlogn)$.

1 Introduction

A polyhedral terrain is a 3-d polyhedral surface such that for each point $v = (x, y, z)$ on the surface, $z = f(x, y)$ for some linear function f. In other words, any vertical line intersects with the terrain at most once. Polyhedral terrains are widely used in computer graphics, computer vision and geography. Some research regarding polyhedral terrain has been done in recent years [Sha88], [CS89].

Sharir [Sha88] posed the problem of computing the shortest watchtower of a given terrain, that is, a shortest vertical line segment which can see every point on the surface. He gave an $O(nlog^2n)$ algorithm, where n is the number of vertices of the terrain, for solving the problem. He also posed the problem of computing the shortest watchtower in $O(nlogn)$ time and made the conjecture that the *fractional cascading* [CG86] or the *hierarchical representation* [DK85] technique might give us the solution.

It turns out that we can solve the problem in $O(nlogn)$ time by using Dobkin-Kirkpatrick's hierarchical representation of a convex polyhedron. In Section 2, we will give the detail of our algorithm. In Section 3, we will discuss the generalized problem of computing the shortest vertical distance between 2 non-intersecting terrains. We will pose a set of closely related open problems for future research in Section 4.

2 Algorithm for the shortest watchtower problem

As pointed in [Sha88], let $f_1, ..., f_n$ be the planar faces of S, and let $\pi_1, ..., \pi_n$ be the planes containing these faces, a point v can see the entire of S if and only if it lies above every π_i. It turn out that this is an unbounded convex polyhedral terrain and can be computed in $O(nlogn)$ time (we denote it as L). Now the problem is reduced to computing the shortest vertical distance between a polyhedral terrain S with n faces and another convex polyhedral terrain L with n faces lying above S. It is easy to see that the shortest vertical line segment uv, with $u \in S$, $v \in L$ must satisfy one of the following properties:

(1) v is a vertex of L;
(2) u is a vertex of S;
(3) u lies on an edge of S and v lies on an edge of L.

As shown in [Sha88], the first two cases can be easily done in a total of $O(nlogn)$ time by applying the planar subdivision method [Kir83]. For the third case, Sharir gave an $O(log^2 n)$ time algorithm for computing the shortest distance between an arbitrary line segment and L. We will improve this bound to $O(logn)$, thus improving the overall bound to $O(nlogn)$. Before we proceed, we make the following definitions. Given two line segments s_1 and s_2 in 3-d, if there is a vertical line l such that $s_1 \cap l \neq \emptyset$, $s_2 \cap l \neq \emptyset$, then the vertical distance between s_1 and s_2 is the difference between the z-cordinate of $s_1 \cap l$ and $s_2 \cap l$. Otherwise the vertical distance between the two segments is infinity.

Let $e = ab$ be an edge of S. For $0 \leq t \leq 1$ define

$$u(t) = (1-t)a + tb \in e,$$

and $v(t)$ is the point on L lying directly above $u(t)$. Let

$$F_e^L(t) = |u(t)v(t)|.$$

We have the following crucial property:

Observation 1: $F_e^L(t)$ is a piecewise linear convex function.

Now we give a brief description of Dobkin-Kirkpatrick's hierarchical representation of a convex polyhedron. Let P be a polyhedron in 3-d with vertex set $V(P)$, edge set $E(P)$ ($|V(P)|, |E(P)| \in O(n)$). A sequence of polyhedra, $H(P) = P_1, ..., P_k$, is said to be a *hierarchical representation* of P if

(i) $P_1 = P$ and P_k is a 3-simplex (i.e, a convex polyhedron whose size is constant);
(ii) $P_{i+1} \subset P_i$, for $1 \leq i < k$;
(iii) $V(P_{i+1}) \subset V(P_i)$; and
(iv) the vertices of $V(P_i) - V(P_{i+1})$ form an independent set (i.e., are non-adjacent) in P_i.

Furthermore, as shown in [DK85], there exist constant $c = 11$ such that for a polyhedron P in 3 dimensions there exists a hierarchical representation (which can be constructed in $O(n)$ time) of degree at most c, $O(logn)$ height, and $O(n)$ size. This immediately implies:

Observation 2: There are at most 11 edges of P_i intersecting any supporting plane of P_{i+1};

Suppose we already have the hierarchical representation $P_k, ..., P_{i+1}, P_i, ..., P_1(P_1 = P)$ for P. We also assume that for P_i the minimum vertical distance between P_i and an arbitrary line segment $\overline{ab}$ is denoted as $min(\overline{ab}, P_i)$ = the minimum vertical distance between $\overline{ab}$ and $\overline{xy}$) over all $\overline{xy} \in P_i$.

Lemma 1: Assume for P_{i+1}, the function $F_e^{P_{i+1}}(t)$ achieves the minimum at the edge $\overline{xy}$ of P_{i+1}, then $min(\overline{ab}, P_i)$ is equal to either $min(\overline{ab}, P_{i+1})$ or the minimum vertical distance between $\overline{ab}$ and $\overline{pq}$ such that $p \in V(P_i) - V(P_{i+1})$, $q \in V(P_{i+1})$, $\overline{pq} \in E(P_i) - E(P_{i+1})$ and $\overline{px}, \overline{py} \in E(P_i) - E(P_{i+1})$, if such p, q exists.

Proof: Note that we have two important properties here: first the distance function $F_e^{P_{i+1}}(t)$ is convex; second, for P_{i+1}, if we add a vertex $p' \in V(P_i) - V(P_{i+1})$, the resulting polyhedron is still convex (adding all these vertices gives us P_i). Now we prove Lemma 1 by contradiction.

Suppose our claim is false, that is, $min(\overline{ab}, P_i) < min(\overline{ab}, P_{i+1})$ and it achieves the minimum at $\overline{p'q'} \in E(P_i) - E(P_{i+1})$ such that $p' \in V(P_i) - V(P_{i+1})$ and $q' \in V(P_{i+1})$ and at least one of $\overline{p'x}$ and $\overline{p'y} \notin E(P_i) - E(P_{i+1})$. We just consider the case when both $\overline{p'x}$ and $\overline{p'y} \notin E(P_i) - E(P_{i+1})$, the other two cases are similar.

Suppose we add only p' (and the corresponding edges adjacent to it) to P_{i+1}, according to the above discussion, the resulting polyhedron P' is still convex. Since we only add p' to P_{i+1} and p' is not incident to either x or y, all the edges incident to x or $y \in E(P_{i+1})$ are also edges in $E(P')$. Consider the distance function between $e = \overline{ab}$ and P', it is easy to see it will have two local minimum, the vertical distance between $\overline{ab}$ and $\overline{xy}$ (this is $min(\overline{ab}, P_{i+1})$, since none of the neighbor edges of $\overline{xy}$ is changed this keeps as a local minimum for the distance function $F_e^{P'}(t)$) and the vertical distance between $\overline{ab}$ and $\overline{p'q'}$

(this is the global minimum by assumption). But this will contradict with the fact that the distance function between a line segment and a convex polyhedron is always convex. □

Lemma 2: If $min(\overline{ab}, P_{i+1})$ is known, to compute $min(\overline{ab}, P_i)$, we need only to go from P_{i+1} to P_i and check at most 22 edges in $E(P_i) - E(P_{i+1})$. Furthermore, this can be done in constant time.

Proof: The first part of Lemma 2 follows immediately from Observation 2 and Lemma 1. The only thing we need to rectify is how to find such $p \in V(P_i) - V(P_{i+1})$ (at most two such p) and search $q \in V(P_{i+1})$ (at most 22) such that $\overline{pq} \in E(P_i) - E(P_{i+1})$ in constant time. We can clarify this when we compute the hierarchical representation of P (without increasing the overall time and space bound for computing the hierarchical representation of P). When we delete $r \in V(P_i)$ to get P_{i+1}, we retriangulate (actually computing the convex hull) of the *"hole"* formed by the deletion of r. For each newly created edge and the boundary edges of the "hole", we assign a parent node (r, i) with it. Note that in this process, no edge can have more than two level-i parent nodes (boundary edge of the hole could have 2). Then when we have $\overline{xy} \in E(P_{i+1})$, we can find its level-$i$ parent nodes in $V(P_i) - V(P_{i+1})$ in $O(1)$ time. From these parent nodes (at most 2), we can list the edges (at most 22) incident to them in $E(P_i)$ in constant time (a DCEL representation of convex polyhedron supports this operation). Then we simply find the shortest vertical distance between these edges and $\overline{ab}$. This is $min(\overline{ab}, P_i)$. □

Lemma 1 and 2 enable us to compute the shortest vertical distance between $e = \overline{ab}$ and P in $O(logn)$ time. We simply start from P_k, each time test at most 22 candidates and keep the current minimum until to get $P_1 = P$, at this stage we have $min(\overline{ab}, P_1)$, which is the solution. The detailed algorithm is trivial and omitted. Now we have the main result of this paper,

Theorem 3: There is an algorithm which computes the shortest watchtower of a polyhedral terrain with n vertices in $O(nlogn)$ time.

3 Compute the shortest vertical distance between 2 non-intersecting polyhedral terrains

As pointed out by Sharir, the obvious generalization is to compute the shortest vertical distance between two arbitrary, non-intersecting polyhedra terrains. Using a technique called *generalized point location*, Chazelle and Sharir obtained an $O(n^{1.999878})$ algorithm for the problem, which beat the trivial $O(n^2)$ time bound [CS90]. We give an $O(nlogn + k)$ time algorithm, where k is the number of intersections of the projections of the edges of the two terrains.

Let R and S be two non-intersecting polyhedral terrains. As shown in Section 2, there are 3 cases for the shortest vertical distance between them. Again we only consider the third case. We use $E(R)$, $E(S)$ to denote the set of edges of R and S respectively. We define a bipartite weighted graph G as follows, $V(G) = E(R) \cup E(S)$, and there is an edge between two segments $s_R \in E(R), s_S \in E(S)$ if and only if their vertical distance is not infinity (if we project the two segments on the XY-plane, they must have an intersection). The weight of such an edge is the vertical distance between them. Note that we can construct G in $O(nlogn + k)$ time by running Chazelle-Edelsbrunner's algorithm [CE88] (The detail is again omitted). Then we can simple find the edge with the minimum weight in $O(k)$ time (there are k edges in G).

Theorem 4: The shortest vertical distance between two nonintersecting terrains can be computed in $O(nlogn + k)$ time.

Although k could be $O(n^2)$, under some circumstances (the expected value of the length of n line segments under uniform distribution is $O(\sqrt{logn/n})$), the expected value of k will be $O(nlogn)$, as shown in a recent paper by Devroye and Zhu [DZ92]. Interested readers should refer to [DZ92] for details.

4 Concluding Remarks

We list some closely related problems for future research as follows:

(1) Can *fractional cascading* be used to solve the shortest watchtower problem?

(2) What is the lower bound of computing the shortest watchtower of a polyhedral terrain? Does the information that L is a special convex polyhedron help improving the $O(nlogn)$ upper bound?

(3) For the general problem we consider in Section 3, is it possible to get a faster algorithm? Note that in 2-d, we can find the shortest vertical distance between 2 non-intersecting monotone chain in $\Theta(n)$ time.

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