On Counting Triangulations in d Dimensions

Tamal Krishna Dey[1]

Abstract

Given a set of n labeled points on S^d, how many combinatorially different geometric triangulations for this point set are there? We show that the logarithm of this number is at most some positive constant times $n^{\lfloor \frac{d}{2} \rfloor + 1}$. Evidence is provided that for even dimenions d the bound can be improved to some constant times $n^{\frac{d}{2}}$.

1 Introduction

In this paper we consider the problem of counting the number of combinatorially different geometric triangulations of a fixed set of n labeled points on S^d, the d-dimensional sphere. By this we mean a triangulation consisting of geometric simplices rather than topological or combinatorial generalizations thereof. A precise definition will be given below. Let $s_d(n)$ denote the maximum number of geometric triangulations with a fixed set P of n labeled points in S^d. A more general type of triangulations often considered in the literature consists of topological simplices in S^d. Let $t_d(n)$ denote the maximum number of topological triangulations of a fixed set of n labeled points in S^d. Every geometric triangulation of S^d is also a topological triangulation. Therefore $s_d(n) \leq t_d(n)$. On the other hand, some of the topological triangulations of P are not realizable geometrically. This is even true if the points can be moved to convenient locations, which is not admitted for the problem considered in this paper.

Using a result of Goodman and Pollack [3], the bounds for a fixed point set can be extended to cover all point sets of some fixed cardinality. More specifically, they show that there is a positive constant $c = c(d)$ so that the logarithm of the number of combinatorially different sets of n points in S^d is at most $cn \log n$. It appears that the dominant factor in the total number of triangulations is the number of triangulations of a single point set rather than the number of different point sets. Kalai [4] proves that for fixed d, the logarithm of the number of topological triangulations for n labeled points (not necessarily fixed) in S^d has a lower bound of $c_1 n^{\lfloor \frac{d}{2} \rfloor}$ and an upper bound of $c_2 n^{\lceil \frac{d}{2} \rceil} \log n$. where c_1 and c_2 are some positive constants. This implies an upper bound of $cn^{\lceil \frac{d}{2} \rceil} \log n$ for $\log s_d(n)$. In general we will use c with or without index for positive constants.

Another quantity related to $s_d(n)$ is $r_d(n)$, the maximum number of geometric triangulations of n fixed and labeled points in $\Re^d$, the d-dimensional real space. It is fairly easy to establish a correspondence between geometric triangulations in S^d and $\Re^d$ that implies $r_d(n) \leq s_d(2n)$, see section 2.

This paper is organized as follows. Section 2 introduces the basic definitions. Section 3 presents an observation about intersecting simplices that is used to prove $\log s_d(n) \leq cn^{\lceil \frac{d}{2} \rceil}$ when d is odd. For even d we generalize a technique inspired by the work of [1] where it was used to prove that

[1]Department of Computer Science, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801, USA and Department of Computer Science, Indiana-Purdue University at Indianapolis, Indianapolis, IN 46202, USA.

$\log s_2(n) \leq cn$. This technique relies on a result that is known to be true in dimension $d = 2$ and which is conjectured to hold for all constant even dimensions. Contingent upon this conjecture, we prove that $\log s_d(n) \leq cn^{\frac{d}{2}}$ for even d.

2 Definitions

Think of S^d as the unit sphere in $\Re^{d+1}$ centered at the origin, o. A ***hemisphere*** of S^d is the intersection of S^d with a closed halfspace in $\Re^{d+1}$ whose bounding hyperplane contains o. Any collection V of k points in S^d is *a.i.* if $V \cup \{o\}$ is affinely independent in $\Re^{d+1}$. V defines a unique ***great sphere*** in S^d, namely the intersection of S^d with the affine hull of $V \cup \{o\}$. If V is a.i. then this great sphere is a $(k-1)$-sphere of S^d. For $0 \leq k \leq d$, a ***spherical polytope*** in S^d is the intersection of finitely many hemispheres. It is a k-polytope if it contains $k+1$ a.i. points but not $k+2$. In what follows, we assume the points in P are in general position. By this we mean that no hemisphere contains P and any $d+1$ points of P are a.i.

A ***spherical k-simplex*** in S^d is the intersection of all hemispheres that contain some set of $k+1 \leq d+1$ points, the vertices of the simplex. Thus, any set V of $k+1 \leq d+1$ a.i. points in S^d defines a unique spherical k-simplex, $\Delta = \Delta_V$. For $0 \leq j \leq k$, a ***j-face*** of Δ is the spherical j-simplex defined by any $j+1$ of the $k+1$ vertices of Δ. Let $\Delta_1 = \Delta_{V_1}$ be a spherical k-simplex and $\Delta_2 = \Delta_{V_2}$ be a spherical ℓ-simplex. We say that Δ_1 and Δ_2 ***intersect improperly*** if $\mathrm{ri}(\Delta_1) \cap \mathrm{ri}(\Delta_2) \neq \emptyset$ where $\mathrm{ri}(X)$ denotes the relative interior of X. If the $k+\ell+2$ vertices in $V_1 \cup V_2$ are a.i. then Δ_1 and Δ_2 intersect improperly iff $\Delta_1 \cap \Delta_2$ is not a face of both. Furthermore, we say that Δ_1 and Δ_2 ***cross*** if they intersect improperly and $V_1 \cap V_2 = \emptyset$. For P a finite set of point in general position in S^d, we denote by $\binom{P}{k}$ the set of all spherical $(k-1)$-simplices with vertices in P. A subset $T \subseteq \binom{P}{k}$ is ***crossing-free*** if no two spherical $(k-1)$-simplices in T cross. A ***geometric triangulation*** of P is defined by a collection of spherical d-simplices Δ_{V_i} so that

(i) $\Delta_{V_i} \cap P = V_i$, for each i,

(ii) no two d-simplices intersect improperly, and

(iii) the union of the d-simplices is S^d.

Conditions (i) and (ii) require that the collection of spherical d-simplices form a simplicial cell complex, and (iii) requires that S^d is the underlying space of the complex.

Similar definitions are possible in $\Re^d$. A set of $k+1 \leq d+1$ affinely independent points defines a unique k-simplex~~, namely the convex hull of the $k+1$ points.~~ Alternatively, this k-simplex can be defined as the intersection of all closed half-spaces that contain the $k+1$ points. A ***geometric triangulation*** of a finite point set $P \subseteq \Re^d$ is defined by a collection of d-simplices so that each d-simplex intersects P in its vertices, no two d-simplices intersect improperly, and the union of the d-simplices is the convex hull of P. By central projection, such a triangulation in $\Re^d$ can be mapped to the northern hemisphere of S^d where it forms a partial triangulation of S^d. Let P' be the set of vertices of this partial triangulation. To complete this triangulation we also project the triangulation from $\Re^d$ to the "southern" hemisphere. Let P'' be the vertex set. The two partial triangulations can be connected by considering the convex hull of $P' \cup P''$ in $\Re^{d+1}$. Any face of the convex hull that has vertices in P' as well as in P'' can now be mapped to a spherical simplex that connects the two partial triangulations. Given the triangulation in $\Re^d$, this construction implies a unique triangulation of S^d. Therefore, $r_d(n) \leq s_d(2n)$.

3 Simplex Crossings in S^d

Given two spherical simplices that intersect improperly, we prove that there is a lower dimensional face of one that crosses a higher dimensional face of the other. In precise, we have the following Lemma which is proved in [2] for simplices in $\Re^d$. In what follows, by a simplex we mean a spherical simplex and by a triangulation we mean a geometric triangulation in S^d.

Lemma 3.1 For $k_1 + k_2 \geq d$, let Δ_1 be a k_1-simplex that intersects improperly a k_2-simplex Δ_2 in S^d. There must be a $\lfloor \frac{d}{2} \rfloor$-face of one simplex that crosses the other simplex.

Proof. Actually one can prove a stronger statement from which the Lemma follows immediately. Let $k_1 + k_2 \geq d$. Then there is an ℓ_1-face of Δ_1 that crosses an ℓ_2-face of Δ_2, with $\ell_1 + \ell_2 = d$. We omit the proof here. For a proof of similar statement in $\Re^d$ see [2]. □

From the above Lemma we have the following simple observation about triangulations in S^d. We observe that all higher dimensional faces of a triangulation can be completely determined from its $\lfloor \frac{d}{2} \rfloor$-faces as follows. To enumerate all k-faces of the triangulation, $k > \lfloor \frac{d}{2} \rfloor$, form all possible k-faces out of the given $\lfloor \frac{d}{2} \rfloor$-faces. Retain only those k-faces that do not intersect any given $\lfloor \frac{d}{2} \rfloor$-face. These are the k-faces of the triangulation. This is true because any k-face of the triangulation must have $\lfloor \frac{d}{2} \rfloor$-faces from the given set of $\lfloor \frac{d}{2} \rfloor$-faces and any k-face that is not in the triangulation must intersect another k-face of the triangulation and hence a $\lfloor \frac{d}{2} \rfloor$-face of the triangulation due to Lemma 3.1.

Lemma 3.2 $\log s_d(n) = O(n^{\lfloor \frac{d}{2} \rfloor + 1})$

Proof. By above observation, any triangulation of n fixed points in S^d can be completely determined by the set of $\lfloor \frac{d}{2} \rfloor$-faces of the triangulation. There can be at most $2^{O(n^{\lfloor \frac{d}{2} \rfloor + 1})}$ different such sets. □

Note that combining Lemma 3.2 with the result of Kalai [4], we get $\log s_d(n) = O(n^{\lceil \frac{d}{2} \rceil})$ for odd dimensions and $\log s_d(n) = O(n^{\lceil \frac{d}{2} \rceil}$ ~~log n) for even dimensions~~. The ~~log~~ n factor in the bound for even dimensions seems unnatural. We show that $\log s_d(n) = O(n^{\frac{d}{2}})$ for even d if we assume the following conjecture. In what follows we assume d is even and $u = \frac{d}{2}$.

Conjecture 3.1 Let T be a set of crossing free u-simplices with n vertices in S^d. Then $|T| = O(n^u)$.

Clearly, $|T| = O(n^{u+1})$, and it is known that $|T| = O(n^u)$ if T forms a subcomplex of a triangulation [5]. Furthermore, a recent result of Živaljević [6] implies that $|T| = O(n^{u+1-\epsilon})$ where $\epsilon = (\frac{1}{3})^u$. Note that since two u-simplices in S^d can intersect only in a point, improper intersection and crossing imply the same thing for them. The following Lemma establishes an important fact about the number of crossings in a set of t u-simplices with n vertices. Let P be a set of n points in S^d and $x^{(d)}(P,T)$ denote the maximum number of u-simplex crossings in a set T of t u-simplices with vertices in P. Define $x^{(d)}(n,t) = \min_{|P|=n,|T|=t}(P,T)$. The next lemma follows from our results in [2] where we proved a stronger version of it.

Lemma 3.3 If the maximum size of any set of crossing free u-simplices with n vertices is $c_1 n^{u+1-\delta}$ (for some constant $0 < \delta \leq 1$) then there exists a constant c_2 so that $x^{(d)}(n,t) \geq c_2\binom{n}{2u+2}\left(\frac{t}{\binom{n}{u+1}}\right)^\gamma$ where $t \geq c_3 n^{u+1-\delta}$, $\gamma = 1 + \frac{u+1}{\delta}$, and $c_3 = c_1 + 1$.

Applying the pigeon-hole principle on the lower bound of $x^{(d)}(n,t)$ we get that there is at least one u-simplex that intersects many other u-simplices. This is stated in the following Lemma.

Lemma 3.4 Let T be a set of t u-simplices in S^d. There exists a u-simplex that crosses $\Omega(\frac{t^{\gamma-1}}{n^{(\gamma-2)(u+1)}})$ other u-simplices where $t > c_3 n^{u+1-\delta}$, and n is the size of the vertex set.

4 Crossing Free Simplices

Using conjecture 3.1 in Lemma 3.4 we get that there exists a u-simplex in T that intersects $\Omega(\frac{t^{u+1}}{n^{u(u+1)}})$ u-simplices. Using this fact we deduce that for even d, there are at most $2^{O(n^u)}$ crossing free sets of u-simplices with n fixed vertices in S^d. Define $F(t)$ as the largest number of crossing free subsets of u-simplices that can be chosen from t u-simplices in S^d with n fixed vertices. Since the set of u-simplices of a triangulation completely determines it, an upper bound on $F(t)$ for $t = \binom{n}{u+1}$ also gives an upper bound on the number of triangulations with n vertices in S^d.

Lemma 4.1 Assuming Conjecture 3.1, $F(t) = 2^{O(n^u)}$ for any even d.

Proof. Let c be large enough so that there is a u-simplex that crosses at least $\frac{(u+1)t^{u+1}}{cn^{u(u+1)}}$ other u-simplices if $t > cn^u \geq c_3 n^u$. Assuming conjecture 3.1, we can always find such a u-simplex due to Lemma 3.4.

Case 1. $t \leq cn^u$.
In this case we have $F(t) \leq 2^t \leq 2^{cn^u}$.

Case 2. $t > cn^u$. In this case we prove that $F(t) \leq C^{n^u} f(t)$ where $C = (2c)^{(c+\frac{1}{c^{u}-1})}$ and $f(t) = (\frac{t}{n^u})^{-\frac{cn^{u(u+1)}}{t^u}}$. We show later that $f(t) \leq 1$ for $c_3 n^u < t < \binom{n}{u+1}$ implying $F(t) = 2^{O(n^u)}$. We use induction.

Base Case: $cn^u \le t \le 2cn^u$.
In this case we have

$$\begin{aligned} F(t) \le 2^{2cn^u} &\le (2c)^{cn^u}\left(\frac{t}{n^u}\right)^{\frac{cn^{u(u+1)}}{t^u}} f(t) \ provided\ c > 2 \\ &\le (2c)^{cn^u}(2c)^{\frac{cn^{u(u+1)}}{c^u n^{u^2}}} f(t) \\ &= (2c)^{(c+\frac{1}{c^{u-1}})n^u} f(t) \le C^{n^u} f(t),\ where\ C = (2c)^{(c+\frac{1}{c^{u-1}})}. \end{aligned}$$

Inductive step: $t \ge 2cn^u$.
Since there is a u-simplex that crosses at least $\frac{(u+1)t^{u+1}}{cn^{u(u+1)}}$ other u-simplices, we have

$$F(t) \le F(t-1) + F\left(t - \frac{(u+1)t^{u+1}}{cn^{u(u+1)}}\right).$$

Let $t = kn^u$ where $2c \le k < n$.

$$\begin{aligned} t - \frac{(u+1)t^{u+1}}{cn^{u(u+1)}} &= kn^u - \frac{(u+1)k^{u+1}n^{u(u+1)}}{cn^{u(u+1)}} \\ &= kn^u\left(1 - \frac{(u+1)k^u}{cn^u}\right) \\ &> kn^u\left(1 - \frac{u+1}{c}\right) > cn^u \ if\ c > 2(u+1). \end{aligned}$$

So we can apply the inductive assumption and get

$$\begin{aligned} F(t) &\le F(t-1) + F\left(t - \frac{(u+1)t^{u+1}}{cn^{u(u+1)}}\right) \\ &< C^{n^u} f(t-1) + C^{n^u} f\left(t - \frac{(u+1)t^{u+1}}{cn^{u(u+1)}}\right) \\ &< C^{n^u} f(t)\ by\ the\ property (5)\ of\ f(t)\ where\ t > 9^{u+1}n^u. \end{aligned}$$

Taking c to be sufficiently large, this proves that $F(t) = 2^{O(n^u)}$ for all $t \ge 0$. □

Now we show that the function f indeed have the properties used in the previous Lemma. Let $f(x) = (\frac{x}{n^u})^{-\frac{cn^{u(u+1)}}{x^u}}$ for $n^u \le x \le \binom{n}{u+1}$ and $c > 0$ is a sufficiently large constant.

(1) $f(x) \le 1$ for $x \ge n^u$.

(2) $f'(x) = f(x)\frac{cn^{u(u+1)}}{x^{u+1}}\{ln(\frac{x}{n^u}) - u\}$. Hence $f'(x) > \frac{cn^{u(u+1)}}{x^{u+1}} f(x)$ if $x > e^{u+1}n^u$.

(3) $f(x) - f(x-1) = f'(y)$ for some $x - 1 \le y \le x$ because of the mean value theorem. Therefore $f(x) - f(x-1) > \frac{cn^{u(u+1)}}{x^{u+1}} f(x-1)$, provided $x - 1 > e^{u+1}n^u$ and hence $f(x-1) < \frac{x^{u+1}}{x^{u+1}+cn^{u(u+1)}} f(x)$.

(4) $f(x - \frac{(u+1)x^{u+1}}{cn^{u(u+1)}}) \leq c' \frac{n^{u(u+1)}}{x^{u+1}} f(x)$ where $c' = (e^{u+1})^{2^u}$ is a constant assuming $c > 2(u+1)$. We omit the proof here.

(5) $f(x-1) + f(x - \frac{(u+1)x^{u+1}}{cn^{u(u+1)}}) < f(x)$ for $x > kn^u$ where k is some constant determined as follows. We have to show that

$$\frac{x^{u+1}}{x^{u+1} + cn^{u(u+1)}} + \frac{c'n^{u(u+1)}}{x^{u+1}} \leq 1$$

Let $x = kn^u$, where $2c \leq k < n$. We show that the above relation can be satisfied for $k > 9^{u+1}$. We must have

$$\begin{aligned} c'cn^{2u(u+1)} + c'x^{u+1}n^{u(u+1)} &\leq cx^{u+1}n^{u(u+1)} \\ c'c &\leq k^{u+1}(c - c') \\ k &> (\frac{c'c}{c-c'})^{\frac{1}{u+1}} \\ &< (2c')^{\frac{1}{u+1}} < 2^{\frac{1}{u+1}} e^{2^u} < 9^{u+1} \; if \; c > 2c'. \end{aligned}$$

Combining the results of Lemma 3.2 and Lemma 4.1 we get the following result.

Theorem 4.2 $\log s_d(n) = O(n^{\lceil \frac{d}{2} \rceil})$ when d is odd. Further, assuming conjecture 3.1, $\log s_d(n) = O(n^{\frac{d}{2}})$ when d is even.

Acknowledgement. The author thanks Herbert Edelsbrunner for many enlightening discussions on this research.

References

[1] M. Ajtai, V. Chvátal, M. M. Newborn and E. Szemerédi. Crossing-free subgraphs. *Ann. Discrete Math.* **12** (1982), 9–12.

[2] T. K. Dey and H. Edelsbrunner. Counting simplex crossings and halving hyperplanes. Manuscript, 1992.

[3] J. E. Goodman and R. Pollack. Upper bounds for configurations and polytopes in $\Re^d$. *Discrete Comput. Geom.* (1986), 219–227.

[4] G. Kalai. Many triangulated spheres. *Discrete Comput. Geom.* **3** (1988), 1–14.

[5] R. Stanley. The upper-bound conjecture and Cohen-Macaulay rings. *Stud. Appl. Math.* **54** (1975), 135–142.

[6] R. Živaljević. New cases of the colored Tverberg's theorem. Personal communications.