# The Space of Spheres, a Geometric Tool to Unify Duality Results on Voronoi Diagrams*

**O. Devillers**[†] **S. Meiser**[‡] **M. Teillaud**[†]

**Abstract:** We show how duality results for generalized Voronoi diagrams can be proven by only using purely geometric interpretations, without any analytic calculations.

## 1 Introduction

[7] presented the first result of duality for Voronoi diagrams. Many geometric transforms for generalized diagrams are now known (see [3] for a very complete survey) and they have found many applications.

We do not claim to give new results, but we introduce a tool allowing to have a global view and, above all, a better understanding of a very large number of duality results.

For example, F. Aurenhammer [2] introduces the same duality as ours, between spheres and points, for power diagrams, but he gives no geometric interpretation to this, and his justifications are only by analytic computations. We can give new proofs, avoiding all analytic calculations, with only geometric reasoning.

We use here a geometric interpretation of spheres as points in the *space of spheres*, which is a very powerful tool, and will probably find many applications in the future, since it allows to deal with some very general problems in a simple way (see for example Section 3.5 for the case of weighted order $k$ power diagrams, or Section 3.6 for Voronoi diagrams of general manifolds). We can also deduce algorithmic application in some cases.

We can apply our framework to different types of Voronoi diagrams, for example the hyperbolic metric (see [6] for the planar case).

It can also be used to solve problems on circles or spheres: for example, the determination of the ring with minimal surface, defined by two cocircular circles, containing a given set of points transforms in the space of spheres into a linear programming problems.

The necessary mathematical background is summarized in the appendix.

[†]INRIA, B.P.93, F-06902 Sophia-Antipolis cedex, France. Phone: +33 93 65 77 62, E-mail: firstname.name@inria.fr.
Part of this work was done during a visit to Max Planck Institut
[‡]Max Planck Institut für Informatik, D-6600 Saarbrücken, Germany. Phone: +49 681 302-5356, E-mail: meiser@mpi-sb.mpg.de

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## 2 The Space of Spheres

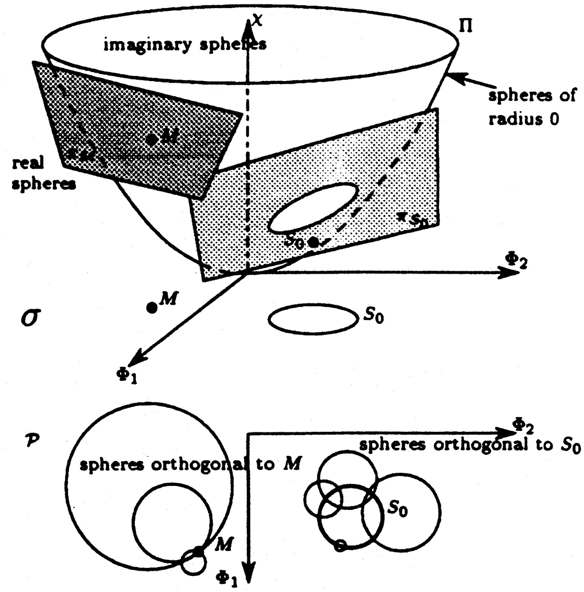


Figure 1: The space of spheres

We denote by $\mathcal{P}$ the euclidean $d$ dimensional space. $\langle .,. \rangle$ is the scalar product in $\mathcal{P}$, and $|..|$ is the euclidean distance between points. A point or a vector $(x_1, \ldots, x_i, \ldots, x_d)$ of $\mathcal{P}$ will be represented as $x$. Let us recall the definition of the *space of spheres* $\sigma$ used in [6]. Let

$$(S) \qquad \langle x,x \rangle - 2\langle x,\Phi \rangle + \chi = 0 \tag{1}$$

be the equation of a sphere[1] $S$ in $\mathcal{P}$. Here $\Phi$ is a point of $\mathcal{P}$, namely the center of $S$. This sphere is represented by the point $S = (\Phi, \chi)$ in the $(d+1)$ dimensional space $\sigma$. $\mathcal{P}$ is identified to the $d$-hyperplane $\chi = 0$ of $\sigma$. Notice that the vertical projection onto $\mathcal{P}$ of a point $S = (\Phi, \chi) \in \sigma$ is the center of the sphere, and the radius $r_S$ of $S$ is given by:

$$\text{power}(O,S) = \chi = \langle \Phi, \Phi \rangle - r_S^2 \tag{2}$$

[1]In the sequel, we will precise the dimension of a geometric object by the following convention: a $p$-object is a manifold of dimension $p$, for example the hyperplanes of $\mathcal{P}$ will be called $(d-1)$-hyperplanes, while hyperplanes of $\sigma$ will be called $d$-hyperplanes. There is only one exception to this convention: if no precision, the word sphere will represent the $(d-1)$-spheres in $\mathcal{P}$, since they are the usual spheres we deal with.

We present in this section the first properties of $\sigma$.

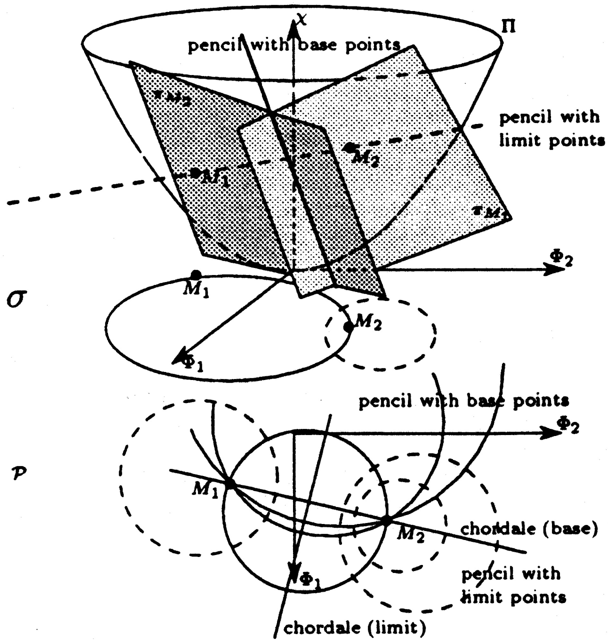


Figure 2: Pencils of spheres with base points, and limit points

## 2.1 The paraboloid Π

Equation (2) shows that the *point-spheres* in $\sigma$ (spheres of radius 0) verify:

$$(\Pi) \qquad \langle \Phi, \Phi \rangle - \chi = 0 \tag{3}$$

Thus, the set of point-spheres is in $\sigma$ the unit $d$-paraboloid of revolution with vertical axis (that is, parallel to the $\chi$-axis) denoted as Π. A point $(\Phi, \chi)$ on Π represents a sphere of $\mathcal{P}$ with center $\Phi$ and radius 0. Therefore points of $\mathcal{P}$ are associated with the corresponding point-spheres on Π in $\sigma$. If space $\mathcal{P}$ is identified to the $d$-hyperplane $\chi = 0$, the point-sphere is obtained by raising the point on Π. The exterior of Π is the set of real spheres, whereas its interior is the set of *imaginary* spheres (with negative square radius) (Figure 1). The square radius of a sphere $S = (\Phi, \chi)$ is the difference of the $\chi$-coordinate of the vertical projection of $S$ on Π with $\chi$.

## 2.2 Hyperplanes in $\sigma$

Two spheres of $\mathcal{P}$ are orthogonal if and only if the two corresponding points of $\sigma$ are conjugate with respect to Π (see Appendix). Thus the set of spheres of $\mathcal{P}$ orthogonal to a given sphere $S_0 = (\Phi_0, \chi_0)$ of $\mathcal{P}$ forms in $\sigma$ exactly the polar $d$-hyperplane $\pi_{S_0}$ of point $S_0$ with respect to Π. Its equation is obtained by polarizing the equation of Π in $S_0$:

$$(\pi_{S_0}) \qquad \chi = 2\langle \Phi_0, \Phi \rangle - \chi_0 \tag{4}$$

It is the polar $d$-hyperplane of the sphere with respect to Π.

$\pi_{S_0} \cap \Pi$ projects in $\mathcal{P}$ onto $S_0$ (Figure 1).

As a particular case, the set of spheres passing through a point $M \in \mathcal{P}$ is also the set of spheres orthogonal to the point-sphere $M$, which is also $\pi_M$, the tangent $d$-hyperplane to Π at $M$. Each sphere in the lower half space limited by $\pi_M$ (i.e. the half space which does not contain Π), contains $M$ in its interior. For a sphere in the upper half space, $M$ is outside (Figure 1).

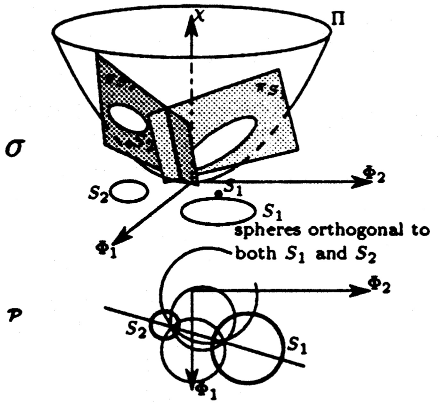


Figure 3: $\pi_{S_1} \cap \pi_{S_2}$

## 2.3 Lines in $\sigma$

A pencil of spheres, that is the set of linear combinations of two spheres of $\mathcal{P}$(see Appendix), transforms in $\sigma$ into the line through the two corresponding points. From Equation (2), a concentric pencil is a vertical line. More generally, a pencil of spheres with limit points is a line hitting Π in the two limit point-spheres (Figure 2). A pencil of spheres with $d$ base points is the common line of the $d$ polar $d$-hyperplanes of the base points (Figure 2). A pencil with tangent point $\Phi$ is a line tangent to Π at the projection of $\Phi$ on Π.

More generally, if $S_1$ and $S_2$ are two spheres, the intersection of their polar $d$-hyperplanes $\pi_{S_1}$ and $\pi_{S_2}$ is the set of all spheres orthogonal to both $S_1$ and $S_2$. All these spheres are necessarily centered on the chordale $\Delta(S_1, S_2)$. Thus $\pi_{S_1} \cap \pi_{S_2}$ projects on $\Delta(S_1, S_2)$ (Figure 3). Lines can be considered as circles with infinite radius, they correspond to points at infinity in $\sigma$.

## 2.4 The map $\rho_k$

We denote by $\rho_k$ the transformation in $\sigma$ that maps a sphere $S = (\Phi, \chi)$ with radius $r$ in $\sigma$ to a sphere $\rho_k(S)$

having the same center and square radius $kr^2$, for $k \in I\!R$. Thus $\rho_k(S) = (\Phi, (1-k)\langle\Phi,\Phi\rangle + k\chi)$

By using the preceding equation, together with Equation (4), we can immediately see that the polar $d$-hyperplane $\pi_{S_0}$ for a sphere $S_0 = (\Phi_0, \chi_0)$ in $\sigma$, maps through $\rho_k$ to the $d$-paraboloid $\rho_k(\pi_{S_0})$ (Figure 4) of equation:

$$(\rho_k(\pi_{S_0})) \qquad \chi = (1-k)\langle\Phi,\Phi\rangle + k(2\langle\Phi_0,\Phi\rangle - \chi_0) \quad (5)$$

We have:

$$\rho_k(\pi_{S_0}) \cap \Pi = \pi_{S_0} \cap \Pi$$

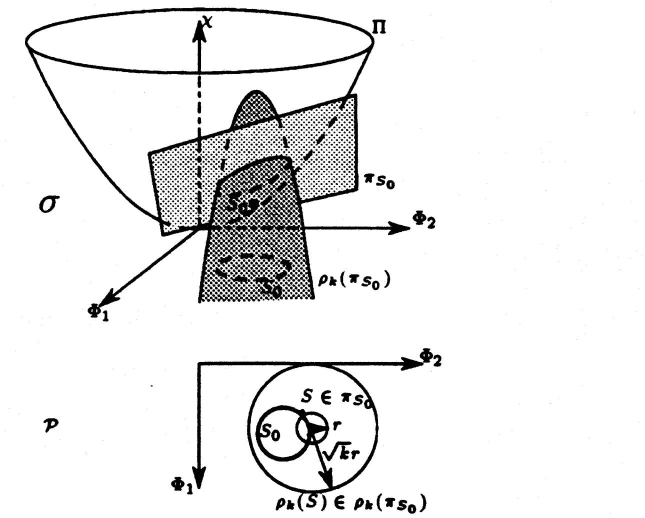


Figure 4: The map $\rho_k$

As a particular case, if $S_0$ is a point-sphere, $\pi_{S_0}$ is the $d$-hyperplane tangent to $\Pi$ at point $(\Phi_0, \langle\Phi_0,\Phi_0\rangle)$, and $\rho_k(\pi_{S_0})$ is also tangent to $\Pi$ at the same point.

# 3 Duality results

We only give a sketch of the results in this abstract, due to lack of space.

We apply our framework to different kinds of Voronoi diagrams. The resulting transformation is often already well known, however, using our framework, the transformation is straightforward and does not require any calculus. Furthermore, our framework allows an easy combination of all possible generalizations of Voronoi diagrams, such as weighted order $k$ power diagrams. It may also lead to new efficient algorithms (see the full paper).

The terms such as distance, or nearest neighbor, refer in each section to the specific distance $\delta$ defined in that particular section. In the sequel, $\mathcal{S}$ always denotes a set of spheres. An element of $\mathcal{S}$ is denoted by $S$ or $S_i$. In some cases, $\mathcal{S}$ is restricted to a special class of spheres, e.g. point-spheres in section 3.1.

## 3.1 Voronoi diagrams of point sites

The distance we consider here is the euclidean distance in $\mathcal{P}$: $\delta(P,Q) = |PQ|$

As noticed in Section 2.2, the set of spheres which do not contain a given point maps in $\sigma$ into the upper half space, limited by the polar $d$-hyperplane of the corresponding point. Thus, if $\mathcal{S}$ is a set of sites in space $\mathcal{P}$, then the set of empty spheres (i.e. the set of spheres which do not surround any site of $\mathcal{S}$) of space $\mathcal{P}$ is in $\sigma$ the intersection of the corresponding half spaces. It is a convex polytope $U_{\mathcal{S}}$, whose facets are tangent to $\Pi$.

For a given point $M \in \mathcal{P}$ and a site $S \in \mathcal{S}$, the intersection of the vertical line through $M$ with $\pi_S$ gives the maximum empty sphere centered at $M$ and touching $S$ (Figure 5). Since the radius of the spheres on the vertical line increases with falling $\chi$, we have:

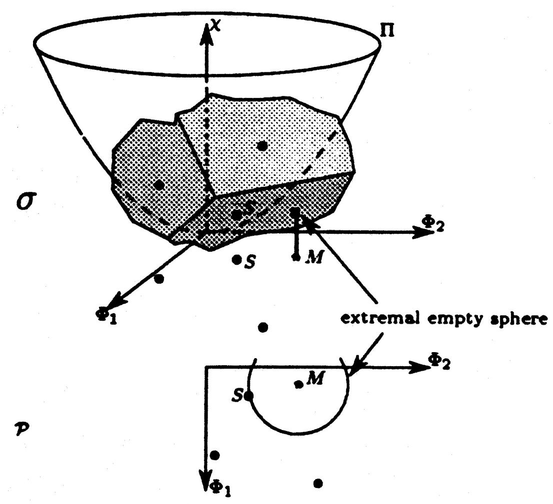


Figure 5: Empty sphere centered on $M$ and touching $S$

*$\mathcal{P}$ 1 The intersection of $U_{\mathcal{S}}$ with a vertical line $\Phi = M$ in $\sigma$ gives the sphere with center $M$ and whose radius is the distance to the nearest neighbor of $M$ in $\mathcal{S}$.*

In other words, the projection of $U_{\mathcal{S}}$ on $\mathcal{P}$ is the Voronoi diagram of $\mathcal{S}$ [8]. This first correspondence in the space of spheres has been used in [6] to compute the Delaunay triangulation of a set of sites lying in $k$ different subspaces, with an output sensitive complexity.

## 3.2 Power diagrams

The power diagram is defined for a set $\mathcal{S}$ of spheres of $\mathcal{P}$. For a sphere $S$ and a point $M$ in $\mathcal{P}$, we define the distance $\delta(M,S)$ from $M$ to $S$ as the power of $M$ with respect to $S$: $\delta(M,S) = \text{power}(M,S)$

In this definition, the spheres of $\mathcal{S}$ can be any spheres, even point-spheres or imaginary spheres. In the case the spheres reduce to points, $\delta$ is the squared usual euclidean distance, and we obtain the usual Voronoi diagram for point sites as a particular case of power diagram.

In space $\mathcal{P}$, the power of $M$ with respect to $S$ can also be seen as the square radius of the sphere which is both centered on $M$ and is orthogonal to $S$. In $\sigma$, this sphere is the intersection of the vertical line passing through $M$ and the polar $d$-hyperplane $\pi_S$. If we now consider all spheres in $\mathcal{S}$, the upper envelope of all their polar $d$-hyperplanes, or equivalently, the intersection of the upper half spaces limited by these $d$-hyperplanes, form a convex polytope $U_{\mathcal{S}}$, and the closest sphere to $M$ for distance $\delta$ is given by the intersection of $U_{\mathcal{S}}$ with the vertical line through $M$.

*$\mathcal{P}$ 2 The intersection of $U_{\mathcal{S}}$ with a vertical line $\Phi = M$ in $\sigma$ gives the sphere with center $M$ and whose square radius is the distance from $M$ to its nearest neighbor in $\mathcal{S}$.*

This shows that the power diagram of the set of spheres $\mathcal{S}$ can be obtained by projecting onto $\mathcal{P}$ the upper envelope of the polar $d$-hyperplanes of the spheres.

The same result has been proved in [2] in a different way, with no geometric interpretation.

This allows to derive an output sensitive algorithm for a constrained set of spheres (see the full paper for a proof):

**Theorem** *Constructing the d dimensional power diagram of n spheres whose centers are constrained to belong to k p-subspaces can be done in time and space*

$$T = O\left(kn^{\gamma+\epsilon}t'^{\gamma+\epsilon} + kt''(\log n)^{O(1)}\right)$$

*for any $\epsilon > 0$. where $t$ is the size the output, $t' = \min(t, n^{\lfloor \frac{p+1}{2} \rfloor})$, $t'' = t - t'$. and*

$$\gamma = \frac{1}{1 + \frac{1}{\lfloor \frac{p+1}{2} \rfloor}}$$

*If $d = 3$ and $p = 2$, the time complexity is $O(tk \log n)$ and the space required is $O(n)$. Furthermore, if $k = 2$, the time complexity reduces to $O(t + n \log n)$.*

## 3.3 Weighted power diagrams

The sites of $\mathcal{S}$ can here be any kind of spheres. Each site $S$ is assigned a weight $w(S)$. The weighted distance of a point to a site is now defined as the power divided by the weight of the site: $\delta(M, S) = \frac{\text{power}(M,S)}{w(S)}$.

Let $S$ denote a site in $\mathcal{S}$ ans let $w(S)$ be its weight. Consider a sphere $S_M = (M, \chi)$, belonging to the paraboloid $\rho_{\frac{1}{w(S)}}(\pi_S)$. $S_M = \rho_{\frac{1}{w(S)}}(S'_M)$ for some sphere $S'_M \in \pi_S$ whose square radius is equal to power$(M, S)$. So the square radius of $S_M$ is $\frac{\text{power}(S,M)}{w(M)} = \delta(M, S)$.

Let us now define $U_{\mathcal{S}}$ as the upper envelope of the set of $d$-paraboloids $\{\rho_{\frac{1}{w(S)}}(\pi_S),\ S \in \mathcal{S}\}$. Intersecting a vertical line through $M$ with $U_{\mathcal{S}}$ gives the site $S$ which minimizes $\delta(M, S)$.

*$\mathcal{P}$ 3 The intersection of $U_{\mathcal{S}}$ with a vertical line $\Phi = M$ in $\sigma$ gives the sphere with center $M$ and whose square radius is the distance from $M$ to its nearest neighbor in $\mathcal{S}$.*

The weighted Voronoi diagram of $\mathcal{S}$ is thus the projection on $\mathcal{P}$ of the upper envelope of the set of $d$-paraboloids $\{\rho_{\frac{1}{w(P)}}(\pi_P), P \in \mathcal{S}\}$.

The first algorithm for computing weighted Voronoi diagrams for point sites in the plane is due to F. Aurenhammer and H. Edelsbrunner [4], and uses a duality that is somewhat different from ours. In [2], there is also a duality with the upper envelope of a set of $d$-paraboloids for the computation of weighted Voronoi diagrams, but the $d$-paraboloids are not the same as ours.

## 3.4 Affine Voronoi diagrams

By ***affine*** Voronoi diagram, we denote Voronoi diagrams whose bisectors are hyperplanes. F. Aurenhammer has shown that every affine Voronoi diagram is a power diagram.

In [2], this result is applied to deduce a duality between weighted Voronoi diagrams for point sites in dimension $d$ and power diagrams in dimension $d+1$. In that way, weighted diagrams reduce to non-weighted diagrams. We give an example in Figure 6, showing that this transformation can nevertheless increase the complexity of the problem. We can show a duality between the weighted diagram in $\mathcal{P}$ and a power diagram in $\sigma$, and we can compute the corresponding sites (see the full paper).

However, in many cases, the complexity does not increase, and this transformation can yield powerful algorithms. For example, if the number of possible weights is smaller than the number of sites, then the output sensitive algorithm stated in Section 3.2 applies to the resulting power diagram.

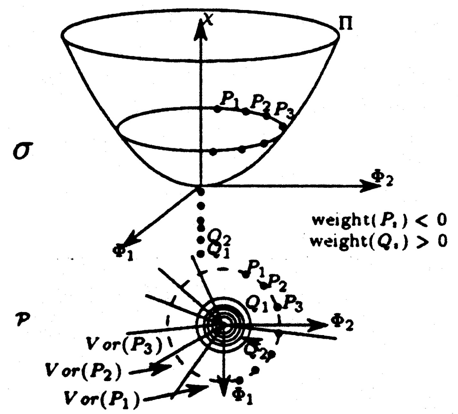


Figure 6: Linear weighted diagram mapped onto a quadratic power diagram

### 3.5 Order $k$ power diagrams

The sites of $\mathcal{S}$ can always be any spheres, and the distance is: $\delta(M,S) = \text{power}(M,S)$

In the order $k$ Voronoi diagram, the regions are associated to subsets of $\mathcal{S}$ of cardinality $k$.

F. Aurenhammer and H. Imai [5,1] use a duality for the computation of these diagrams. Our interpretation allows to give an easy proof. The following relation, for $k$ elements $S_1,\ldots,S_k \in \mathcal{S}$ is given in the appendix:

$$\frac{1}{k}\sum_{i=1}^{k}\text{power}(M,S_i) = \text{power}\left(M, \frac{1}{k}\sum_{i=1}^{k} S_i\right)$$

Let $\mathcal{S}_k$ denote the set of the centers of mass in $\sigma$ of all possible $k$-tuplesof spheres of $\mathcal{S}$. It follows that the spheres $S_1,\ldots,S_i,\ldots,S_k$ are the $k$ nearest neighbors of $M$ if and only if the power of $M$ with respect to their center of mass is smaller than the power of $M$ with respect to the other elements of $\mathcal{S}_k$. This means exactly that the order $k$ power diagram of a set $\mathcal{S}$ of spheres is the usual power diagram of $\mathcal{S}_k$. Using ($\mathcal{P}$ *2*), we deduce:

*$\mathcal{P}$ 5 The intersection of $U_{\mathcal{S}_k}$ with a vertical line $\Phi = M$ in $\sigma$ gives the sphere with center $M$ and whose square radius is the average of the powers of $M$ with respect to the $k$ nearest neighbors of $M$ in $\mathcal{S}$.*

#### Weighted order $k$ power diagram

The powerful tools provided by the space of spheres allow to easily combine weighted diagrams with order $k$ power diagrams, which could not be done in a straightforward manner with the methods previously used in the literature. We can show that the weighted order $k$ Voronoi diagram of a set $\mathcal{S}$ of spheres is the weighted diagram of the centroids of all $k$-tuples of sites of $\mathcal{S}$ (in the computation of the centroids, each sphere $S \in \mathcal{S}$ is associated with a coefficient $\frac{1}{w(S)}$).

### 3.6 Voronoi diagrams of general manifolds

We can now study some very general problems. The elements of $\mathcal{S}$ are now manifolds of any dimensions immersed in $\mathcal{P}$. The distance is the usual distance from a point $M$ to a manifold $Z$ ($|MP|$ is the euclidean distance between $M$ and $P$):

$$\delta(M,Z) = \min_{P\in Z}|MP|$$

In other words, $\delta(M,Z)$ is the radius of the minimum sphere centered at $M$ and tangent to $Z$. This sphere can be obtained as the intersection in $\sigma$ of the vertical line $\Phi = M$ with the set $\Gamma_Z$ of all spheres tangent to $Z$.

$\Gamma_Z$ is a manifold in $\sigma$, it is the upper envelope of the polar planes of the point-spheres of $Z$. If $Z$ is an analytic manifold in $\mathcal{P}$, then $\Gamma_Z$ is an analytic manifold in $\sigma$. We consider the upper envelope $U_{\mathcal{S}}$ of the manifolds $\Gamma_Z$ for all $Z \in \mathcal{S}$.

*$\mathcal{P}$ 6 The intersection of $U_{\mathcal{S}}$ with a vertical line $\Phi = M$ in $\sigma$ gives the sphere with center $M$ and whose radius is the distance to the nearest neighbor of $M$ in $\mathcal{S}$.*

This can be applied to hyperplanes, portions of hyperplanes (Figure 7), spheres, portions of spheres...

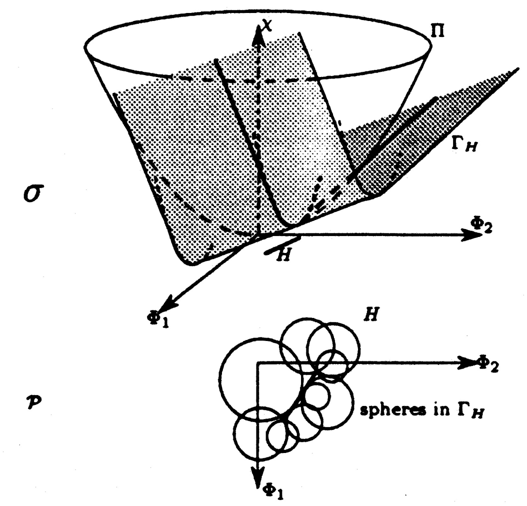


Figure 7: The manifold $\Gamma_H$ for a $(d-1)$-hyperplane or a portion of a $(d-1)$-hyperplane


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## A Mathematical prerequisites

We only recall here some definitions and properties, without any proof.

Let $I\!E^d$ be the $d$ dimensional euclidean space and $\langle .,. \rangle$ the scalar product in $I\!E^d$.

### Circles and spheres

We assume all the equations of the spheres to be given in an orthonormal basis, and to be normalized.

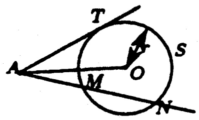


Figure i: Power of a point with respect to a sphere

If $S$ is a sphere with center $O$ and radius $r$, and $A$ is a point, the *power* of $A$ with respect to $S$, power$(A,S)$, is defined equivalently as (Figure i):

$$\begin{aligned} \text{power}(A,S) &= \left\langle \overrightarrow{AM}, \overrightarrow{AN} \right\rangle && \text{where } (AMN) \text{ is any line intersecting } S \text{ at } M \text{ and } N \\ &= \left\langle \overrightarrow{AT}, \overrightarrow{AT} \right\rangle && \text{if } A \text{ is exterior to } S \text{ and } (AT) \text{ is a line tangent to } S \text{ at } T \\ &= \left\langle \overrightarrow{AO}, \overrightarrow{AO} \right\rangle - r^2 \end{aligned}$$

The last definition shows that power$(A,M) > 0$ if and only if $A$ is exterior to $S$.

If $S$ has normalized equation

$$S(M) = \sum_{i=1}^{d} x_i^2 + \sum_{i=1}^{d} a_i x_i + a_0 = 0$$

we have: power$(A,S) = S(A)$. This implies the following relation:

$$\sum_{i=1}^{k} \frac{\text{power}(A,S_i)}{w(S_i)} = \left( \sum_{i=1}^{k} \frac{1}{w(S_i)} \right) \text{power}\left( A, \frac{\sum_{i=1}^{k} \frac{S_i}{w(S_i)}}{\sum_{i=1}^{k} \frac{1}{w(S_i)}} \right)$$

where we denote as $\sum_{i=1}^{k} \lambda_i S_i$ the sphere having as equation the corresponding linear combination of the equations of spheres $\{S_i,\ i = 1, \ldots, k\}$, and as $w(S_i)$ a weight associated to each sphere $S_i$.

Let us give another useful property: power$(A,S)$ is the square radius of the sphere centered on $A$ and orthogonal to $S$.

The locus of a point that has the same power with respect to 2 spheres $S_1$ and $S_2$ is an hyperplane called the *chordale* of the spheres and denoted as $\Delta(S_1,S_2)$ (in the plane, it is a line called the *radical axis*). The equation of $\Delta(S_1,S_2)$ is obtained by substracting the two equations of $S_1$ and $S_2$.

A *pencil* of spheres is a linear family of spheres, i.e. the equations of the spheres of the pencil are linearly dependant of a parameter. All pairs of spheres in the pencil have the same chordale. A pencil of spheres has three equivalent definitions: it is a linear family of spheres generated by two given spheres, or the set of spheres that are orthogonal to $d$ given spheres; if $S_1$ and $S_2$ are two given spheres, the pencil defined by $S_1$ and $S_2$ is also the set of spheres that have the same chordale with $S_1$ than $S_2$. The different kinds of pencils will be detailed later.

### Polarity

Four points $M, N, A, B$ lying on a common line are said to form an *harmonic division* if $\frac{MA}{MB} = \frac{NA}{NB}$ and we write in this case $(M,N,A,B) = \frac{\overline{MA}}{\overline{MB}} \cdot \frac{\overline{NB}}{\overline{NA}} = -1$

We can remark that if $(M,N,A,B) = -1$, the sphere of diameter $AB$ is orthogonal to any sphere passing through $M$ and $N$ (and symmetrically, the sphere of diameter $MN$ is orthogonal to any sphere passing through $A$ and $B$) (Figure ii).

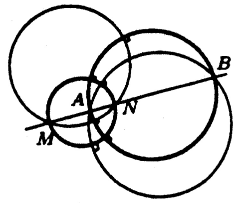


Figure ii: Harmonic division

$M$ and $N$ are said to be *conjugate* with respect to a quadric $Q$ if $(M,N,A,B) = -1$ where $\{A,B\} = (MN) \cap Q$ (Figure iii). If $M$ is exterior to $Q$, and if we choose for the line

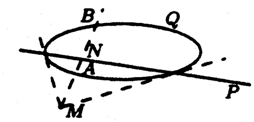


Figure iii: Conjugate points and polar hyperplane

$(MN)$ a tangent from $M$ to $Q$, then $A = B = N$ is the tangent point. The locus of the conjugate of $M$ with respect to $Q$ is a hyperplane $P$ passing through these tangent points.

In the projective space associated to $I\!E^d$, the projective quadric associated to $Q$ is the kernel of a quadratic form $q$. In fact, the conjugation with respect to $Q$ is nothing else that the orthogonality defined in the projective space by $q$;

If the equation of $Q$ is:

$$\sum_{1 \le i \le j \le d} a_{ij} x_i x_j + \sum_{i=1}^{d} a_{0i} x_i + a_{00} = 0$$

then the equation of the polar hyperplane $P$ of $M$ with respect to $Q$ can be obtained by *polarizing* in $M$ the equation of $Q$:

$$\sum_{i=1}^{d} a_{ii} m_i x_i + \sum_{1 \le i < j \le d} \frac{a_{ij}}{2}(m_i x_j + m_j x_i) + \sum_{i=1}^{d} \frac{a_{0i}}{2}(x_i + m_i) + a_{00} = 0$$

where $M = (m_1, \ldots, m_d)$.