Geometry Based Approximations for Intersection Graphs
(Extended Abstract)

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1 Introduction, Motivation and Summary of Results

Intersection graphs of geometric objects have been both widely studied and used to model many problems in real life [Ro78]. In this paper we consider intersection graphs of regular polygons, emphasizing intersection graphs of unit disks. A graph is a **unit disk graph** iff its vertices can be put in a one to one correspondence with equisized circles in a plane in such a way that two vertices are joined by an edge iff the corresponding circles intersect. (It is assumed that tangent circles intersect.) These graphs have been used to model broadcast networks [Ha80, Ka84, YWS84] and optimal facility location [WK88]. For example, the problem of placing k facilities where proximity is undesirable can be modeled as the problem of finding an independent set of size k in a unit disk graph [WK88]. The problem of assigning distinct frequencies for transmitters with intersecting ranges corresponds to the minimum coloring problem for unit disk graphs. The minimum dominating set problem corresponds to selecting a minimum number of transmitters so that all the other stations are within the range of at least one of the chosen transmitters.

All of the above problems and many others remain NP-hard for unit disk graphs [CCJ90]. Motivated by the practical importance of these problems, we present fast heuristics with good performance guarantees for a number of such problems including the three mentioned above. Our heuristics are based on simple geometric properties of unit disk graphs. It is easy to see that similar properties hold for the intersection graphs of other regular polygons and geometric objects in higher dimensions. Consequently, our heuristics can be extended to these graphs as well.

Our results are summarized as follows:

1. The minimum weighted vertex cover problem can be approximated to within a factor of 1.5 of the optimal value. (The best known heuristic for general graphs guarantees a vertex cover which is within a factor of 2 of the optimal [GJ79, YE85].)

2. The maximum independent set problem for unit disk graphs can be approximated to within a factor of 3 of the optimal value.

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3. The minimum vertex coloring problem can be approximated to within a factor of 9 of the optimal value for unit disk graphs. In this case, the performance guarantee remains 9 for all polygons. (Interestingly, our coloring heuristic is also an *on-line* heuristic [Ira91], and hence we obtain a 9-competitive algorithm for on-line coloring of unit disk graphs. On-line coloring algorithms with constant competitive ratios have been obtained previously for several other classes of graphs [GL88, Sl87, KT81].)

4. We present a *simple* heuristic to approximate the minimum degree Steiner tree to within an *additive* constant of 4. (For this problem, the best known heuristic for general graphs guarantees a difference of $O(\log n)$ with respect to the optimal value [FR92].)

5. The minimum dominating set problem for unit disk graphs can be approximated to within a factor of 5 of the optimal value. The same heuristic also provides a performance guarantee of 5 for the minimum independent domination problem. This heuristic can be modified to obtain another heuristic which has a performance guarantee of 10 for both the minimum total domination and the minimum connected domination problems. (Whether constant performance guarantees are obtainable in polynomial time for the dominating set problem for general graphs is a well known open question. Irwing [Ir91] has shown that that the minimum independent domination problem cannot be approximated to within a constant factor unless $\mathbf{P} = \mathbf{NP}$.)

6. For intersection graphs of isothetic unit squares, the maximum independent set and the minimum coloring problems can be approximated to within a factor of 2. (These heuristics exploit the fact that the corresponding optimization problems can be solved efficiently for interval graphs [Go80].)

The remainder of the paper is organized as follows. Section 2 contains definitions and preliminaries. Section 3 discusses some of the approximation algorithms. Section 4 contains concluding remarks. Due to space limitations, only a few proof sketches are included. Detailed proofs will appear in the full paper.

2 Definitions and Preliminaries

The definition of unit disk graphs given above, is referred to as the **intersection model** [CCJ90]. In addition, we will also use the **proximity model** [CCJ90] in which the nodes of the graph are in one-to-one correspondence with a set of points in the plane, and two vertices are joined by an edge iff the distance between the corresponding points is at most some specified bound.

In the intersection model, we assume that the radius of each disk is r. It is easily seen that two such disks in the plane intersect iff the distance between their centers is at most $2r$. Thus one can translate a description of a given unit disk graph in the intersection model into a description in the proximity model and vice versa in linear time. throughout this paper, we assume that a unit disk graph is specified using one of the above models. (We note that it is open whether unit disk graphs can be recognized in polynomial time [CCJ90].)

We assume that the reader is familiar with the definitions of standard graph theoretic optimization problems such as minimum vertex cover, maximum independent set, minimum coloring and minimum dominating set as given in [GJ79].

We use $K_{p,q}$ to denote the complete bipartite graph with p and q nodes in the two sets of the bipartition. We also use K_r to denote the clique with r nodes. Given a graph $G(V, E)$ and a node v, we use $N(v)$ to denote the set of nodes adjacent to v; we refer to $N(v)$ as the **neighborhood** of v. For a graph $G(V, E)$ and $V' \subseteq V$, $G(V')$ denotes the vertex induced subgraph of G. Henceforth, we assume that $|V| = n$.

Most of our heuristics are based on a forbidden subgraph property of unit disk graphs. The proof of this property relies on a geometric observation concerning packing of unit disks in the plane. (The problems of packing and covering by geometric objects have been of interest to researchers for some time. See [CS88] for more on this subject.)

Lemma 2.1 *Let C be a circle of radius r and let S be a set circles of radius r such that every circle in S intersects C and no two circles in S intersect each other. Then, $|S| \leq 5$.*

Sketch of Proof: Suppose $|S| \geq 6$. Let s_i, $1 \leq i \leq 6$, denote the centers of any six circles in S. Let c denote the center of C. Denote the ray $\overrightarrow{c\,s_i}$ by r_i $(1 \leq i \leq 6)$. Since there are six rays emanating from c, there must at least one pair of rays r_j and r_k such that the angle between them is at most 60°. Now, it can be verified that the distance between s_j and s_k is at most $2r$, which implies that circles centered at s_j and s_k intersect, contradicting our assumption. Thus $|S| \leq 5$. ■

In [CS88] a similar parameter called the *kissing number* τ_d is defined as the maximum number of non-overlapping unit balls in $\mathbf{R}^d$ that can be arranged so that all of them touch a central ball. It is well known that $\tau_1 = 1$ and $\tau_2 = 6$. The known bounds for higher dimensions are not quite as tight. Lemma 2.1 also follows from the fact that $\tau_2 = 6$. An immediate consequence of the lemma is the following.

Lemma 2.2 *Let $G(V, E)$ be a unit disk graph. Then G cannot contain an induced subgraph isomorphic to $K_{1,6}$ (i.e., G is claw free for claws of size 6).* ■

Lemma 2.2 implies that in any unit disk graph, the size of a maximum independent set in the subgraph of G induced on the neighborhood of any vertex is at most 5. The next lemma, proven in a manner similar to that of Lemma 2.1, shows that the neighborhoods of certain nodes in a unit disk graph have even smaller independent sets.

Lemma 2.3 *Let G be a unit disk graph, and let v be a vertex such that the center of the unit disk corresponding to v has the smallest X-coordinate. The size of a maximum independent set in the subgraph induced by $N(v)$ is at most 3.* ■

Another combinatorial observation about unit disk graphs is stated in the following lemma. The proof is similar to that of Lemma 2.1 and is omitted due to lack of space.

Lemma 2.4 *Let $G(V, E)$ be a unit disk graph. Then G cannot contain an induced subgraph isomorphic to $K_{2,3}$.* ■

3 Approximations

To illustrate our ideas, we discuss two heuristics, one to approximate maximum independent set and the other to approximate minimum coloring. The approximation algorithms are presented for unit disk graphs. Comments indicating how they can be extended to intersection graphs of other regular polygons are included. For simplicity, we often do not distinguish between a vertex of a unit disk graph and its corresponding disk.

3.1 Approximating Maximum Independent Set

Since unit disk graphs are $K_{1,6}$ free, one can use the algorithm in [Ho85] to obtain a bound of 5. However, we can do better for unit disk graphs because of the additional geometric structure they possess.

The approximation algorithm (which we call IS) begins by forming a list L of the centers of the unit disks in increasing order of their X-coordinates. At each step, the heuristic adds the first item v from L into the indepedent set and deletes from L both v and the nodes which are adjacent to v. The algorithm terminates when L becomes empty. The following theorem, which gives the performance bound of this heuristic, is a direct consequence of Lemma 2.3.

Theorem 3.1 *Let $IS(G)$ and $OPT(G)$ denote respectively the approximate independent set produced by* IS *and an optimal independent set for a unit disk graph G. Then $|IS(G)| \geq |OPT(G)|/3$.* ■

Similar approximations can be given for other regular polygons and objects in higher dimensions. Of course, the performance guarantee is a function of the number of sides of the polygon.

3.2 Approximating Minimum Coloring

The idea here is to divide the plane $\mathbf{R}^2$ into small non-overlapping square cells (called **grid cells**) of equal size. Each grid cell can thus be assigned integer coordinates (i, j), where the grid cell containing the origin has coordinates $(0, 0)$. Knowing the size of a grid cell, we can easily compute the coordinates of the grid cell containing any given point in $\mathbf{R}^2$ (assuming that grid cells are closed). Let r be the radius of the unit disks under consideration. We choose the grid cell to be a square of side $\sqrt{2}r$. By doing so, we ensure that all the circles whose centers lie in the same grid cell form clique. This helps us to get a lower bound on the number of colors used in an optimal coloring. Further, it is also easy to optimally color the subgraph corresponding to each grid cell.

Once we complete the mapping of the circles to grid cells, it is easy to compute the maximum number (say l) of points in any grid cell. Thus the graph has a clique of size l, and so at least l colors are necessary in any valid coloring. We can obtain a valid coloring using at most $9l$ colors as follows. The $9l$ colors are partitioned into 9 color classes with l colors each. We denote these color classes by C_{ij}, where $0 \leq i, j \leq 2$. We group the grid cells into disjoint **tiles**, where each tile is a 3×3 array of subcells. For vertices within each tile, we use the above set of $9l$ colors. Given a unit disk c, let (i_c, j_c) denote the grid cell containing the center of c. The color used for c is chosen from the color class C_{ij}, where $i = i_c \bmod 3$ and $j = j_c \bmod 3$. Since each color class has l colors and each grid cell has at most l centers, we can assign a distinct color to each disk within a grid cell. Hence we obtain the following.

Theorem 3.2 *The number of colors used by the grid heuristic is no more than 9 times the colors used by any optimal algorithm.* ■

We observe that the above heuristic provides a performance guarantee of 9 even when the unit disks are given one at a time (i.e., under the on-line model). For intersection graphs of other regular polygons, the size of the grid cell can be appropriately chosen so that the bound remains 9.

4 Concluding Remarks

We have sketched efficient approximations for two standard problems on unit disk graphs. Our heuristic for vertex cover uses ideas from [YE85, YE82]. Heuristics for domination problems rely on Lemma 2.1 and the heuristic for the minimum degree Steiner tree problem uses depth-first-search. These heuristics can be modified easily to obtain similar bounds for intersection graphs of other regular polygons and for geometric objects in higher dimensions. In [Ho85] efficient approximations were given for the weighted vertex cover and independent set problems. Although these heuristics are applicable to unit disk graphs, our methods exploit the geometric structure of the underlying graph and yield better performance guarantees for most of the problems considered there.

In [HM85] approximation schemes were presented for packing and covering problems for a set of unit disks. These algorithms were similar in flavor to those of [Ba83] in which polynomial time approximation schemes were given to solve various graph problems for planar graphs. It would be interesting to see if one can use a similar approach to design good approximation schemes for the problems considered here.

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References

[Ba83] B.S. Baker, "Approximation Algorithms for NP-complete problems on Planar Graphs," *Proc. 24th FOCS.*, 1983, pp 265-273.

[CCJ90] B.N. Clark, C.J. Colbourn, and D.S. Johnson, "Unit Disk Graphs," *Discrete Math.*, Vol. 86, 1990, pp. 165-177.

[CS88] J.H. Conway and N.J.A. Sloane *Sphere packings, Lattices and Groups,* Springer Verlag, 1988.

[FR92] M. Furer and B. Raghavachari, "Approximating the Minimum Degree Spanning Tree to Within One from the Optimal Degree," *Proc. 3rd SODA*, Jan. 1992, pp 317-324.

[GJ79] M.R. Garey and D.S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-completeness,* W.H. Freeman and Co., San Francisco, CA, 1979.

[Go80] M.C. Golumbic, *Algorithmic Graph Theory and Perfect Graphs,* Academic Press, New York, 1980.

[GL88] A. Gyarfas and J. Lehel, "On-line and First-Fit Colorings of Graphs," *J. Graph Theory,* Vol. 12, No.2, 1988, 217-227.

[Ha80] W.K. Hale, "Frequency Assignment: Theory and Applications," *Proc. IEEE,* Vol. 68, 1980, pp 1497-1514.

[Ho85] Dorit S. Hochbaum, " Efficient Bounds for the Stable Set, Vertex Cover and Set Packing Problems," *Discrete Applied Mathematics,* No. 6, 1983, pp 243-254.

[HM85] Dorit S. Hochbaum, Wolfgang Mass, "Approximation Schemes for Covering and Packing Problems in Image Processing and VLSI," *JACM,* Vol 32,No. 1, 1985, pp 130-136.

[Ira91] S. Irani, "Coloring Inductive Graphs On-Line," *Proc. 30th FOCS,* 1990, pp 470-479.

[Ir91] R.W. Irving, "On Approximating the Minimum Independent Dominating Set," *Inf. Proc. Lett.*, Vol. 37, 1991, pp 197-200.

[Ka84] K. Kammerlander, "C 900 – An Advanced Mobile Radio Telephone System with Optimum Frequency Utilization," *IEEE Trans. Selected Areas in Communication,* Vol. 2, 1984, pp 589-597.

[KT81] H.A. Kierstead and W.A. Trotter, "An Extremal Problem in Recursive Combinatorics," *Congressus Numerantium,* Vol. 33, 1981, pp 143-153.

[Ro78] Fred S. Roberts, *Graph Theory and and Its Applications to Problems of Society,* SIAM, CBMS-NSF Regional Conf. Series in Applied Mathematics, Philadelphia, 1978.

[Sl87] M. Slusarek, "A Coloring Algorithm for Interval Graphs," *Proc. 1987 MFCS.*, Lecture Notes in CS, Vol. 379, 1987, pp 471-480.

[YWS84] Y. Yeh, J. Wilson and S.C. Schwartz, "Outage Probability in Mobile Telephony with Directive Antennas and Macrodiversity," *IEEE Trans. Selected Areas in Communication,* Vol. 2, 1984, pp 507-511.

[WK88] D.W. Wang and Y.S. Kuo, "A Study of Two Geometric Location Problems," *Inf. Proc. Letters,* Vol. 28, No. 6, Aug. 1988, pp 281-286.

[YE82] R. Bar-Yehuda, S. Even, "On Approximating a Vertex Cover for Planar Graphs," *Proc. 14th Annual ACM Symposium on Theory of Computing,* May, 1982. pp 303-309.

[YE85] R. Bar-Yehuda and S. Even, "A local Ratio Theorem for Approximating the weighted vertex cover problem," *Analysis and Design of Algorithms for Combinatorial Problems,* in the Annals of Discrete Mathematics, North Holland Publications. Vol. 25, 1985, pp 27-45.